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GAUSSIAN RANDOM PROCESSES. PART 1,(U)

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Part 1 of 2

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GAUSSIAN RANDOM PROCESSES

by

I. A. Ibragimov, Yu. A. Rozanov

FOREIGN TECHNOLOGY DIVISION

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

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In the book are examined some urgent problems of the theory of the random processes, in development of which a great role is played by the works of authors themselves. It is calculated first of all to the specialists in the probability theory, but its many sections they are of interest for the theory of complex variable functions and a functional analysis. Some sections of a book directly concern important applied problems of the type of the "isolation/liberation of signal against the background of random noise", etc.

Pages 3-4. No typing.

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PREFACE

The book is dedicated in essence to the following three problems which are examined by us in the case of Gaussian stationary processes. First, this explanation of the conditions of the mutual absolute continuity (equivalency) of the different probability distributions of the "segment of random process" and the determination of efficient formulas for the densities of equivalent distributions. In the second place, the description of the classes of

the spectral measures, which correspond in this or another sense to regular stationary processes (specifically, that satisfy the known condition of "powerful mixing"), and also the sub-classes, which correspond that or to another "speed of mixing". Thirdly, this evaluation of the unknown average value of the random process, minus of this average, which is stationary, i.e., figuratively speaking, the problem of the "isolation/liberation of signal against the background of steady noise". Furthermore, in the book are stated some auxiliary information (distribution in hilbert spaces, different properties of sample distribution functions, a series of the theorems of the theory of the complex variable functions, etc.).

The problem about the equivalency of different infinite-dimensional Gaussian distributions, beginning approximately from 1958, intensely studied by many mathematicians (systematic presentation of the basic results can be found, for example, in monograph [23]). In the book consecutively is examined the case of Gaussian stationary processes, in which, as we set/assume, it was possible to obtain the sufficiently final solution.

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The second of the problems indicated closely related with the questions of the ergodic theory of Gaussian dynamic systems, with the

theory of forecasting stationary processes and concerns (from probabilistic point of view) the conditions of the weak dependence of "future" process on its "past" whose use led to so fruitful a theory of limit theorems for dependent variables (for example, see [14], [22]); by the most widely known condition of such kind, apparently, is the so-called condition of "powerful mixing". The problems, which appear in the examination of the varied conditions of regularity, in the Gaussian case are reduced to the peculiar approximating problems of linear spectral theory. The research in this direction led to the almost final solution to problem. In the book will be fed the result of these research.

Finally, the problem of the evaluation of average is, possibly, oldest and most popular in mathematical statistics. There are two known approaches to the solution to this problem. Specifically,, by knowing the spectral density of steady noise, it is possible to construct the best unbiased estimates; otherwise it is possible to resort to the method of least squares.

We propose one general class of the estimations, named "pseudobest", which includes and the classical estimations of least squares, and the best unbiased estimates. For such "pseudobest" estimations are given explicit expressions, are located the conditions of justifiability, are derive/concluded asymptotic

formulas for a correlation error matrix, are establish/installed the conditions of asymptotic efficiency.

One should say that the results which concern the conditions of regularity and evaluation of average, are formulated in spectral terms and automatically are transferred (within the framework of "linear theory") by arbitrary stationary in the broad sense processes.

In conclusion we communicate that the numbering of formulas, theorems and so forth its in each chapter. For example, (4.21) it indicates formula 21 of §4 current chapters. For reader's convenience in text frequently are given the references to known textbooks and monographs (as exception/elimination - the review papers) whose copy is given at the end of the book.

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Chapter I

INTRODUCTION. SOME PRELIMINARY INFORMATION.

§1. Gaussian probability distributions in Euclidian space.

Probability distribution P in N-dimensional vector space R^n is called Gaussian, if the characteristic function

$$\varphi(u)=\int_{R^n} e^{i(u,x)}P(dx), \quad u\in R^n$$

(here $(u,x)=\sum u_k x_k$ indicates the bared vector product $u=(u_1, \ldots, u_n)$ and $x=(x_1, \ldots, x_n)$) takes the form

$$\varphi(u)=\exp\left\{i(u,a)-\frac{1}{2}(Bu,u)\right\}, \quad u\in R^n, \qquad (1.1)$$

where $a=(a_1, \ldots, a_n)\in R^n$ - the so-called average value, B is a linear positive operator, called correlation operator; his assigning matrix/die $\{B_{kl}\}$ is called correlation matrix/die. In this case

$$(u, a) = \int_{R^n} (u, x) \mathrm{P}(dx),$$

$$(Bu, v) = \int_{R^n} [(u, x) - (u, a)] [(v, x) - (v, a)] \mathrm{P}(dx); \quad (1.2)$$

$$u, v \in R^n.$$

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Probability distribution P with the average value of a and the correlation operator B is concentrated in the m-dimensional hyperplane L of space R^n (m-rank of the correlation matrix/die), which can be described as

$$L = a + BR^n.$$

(L is a set of all vectors $y \in R^n$ form y = a + Bx, $x \in R^n$). Specifically,

$$\mathrm{P}(R^n \setminus L) = 0,$$

whereupon probability distribution P is absolutely continuous relative to Lebesgue measure dy in hyperplane L:

$$\mathrm{P}(\Gamma) = \int_{\Gamma \cap L} p(y)\, dy, \quad (1.3)$$

where the density of distribution $p(y), y \in L$, has the form

$$p(y) = \frac{1}{(2\pi)^{m/2} \det B} \exp\left\{-\frac{1}{2}(B^{-1}(y-a), (y-a))\right\} \quad (1.4)$$

(here det B indicates the determinant of the matrix/die, which assigns operator B in subspace $R^m = BR^n$, and B^{-1} is an operator in this subspace, reverse to B).

§2. Gaussian random functions. Assignment of probability measure.

Let $(\Omega, \mathfrak{A}, P)$ be certain probability space, i.e., the measurable space of cell/elements $\omega \in \Omega$ with probability measure P in certain σ-algebra $\mathfrak{A}$ sets $A \subseteq \Omega$.

The real measurable function $\xi = \xi(\omega)$ on space Ω is called random variable. The set of random variables $\xi(t) = \xi(\omega, t)$ (parameter t passes certain multitude T) is called the random function of the parameter $t \in T$. Random variables themselves $\xi(t)$ are called the values of this random function $\xi = \xi(t)$; with that which was fix/recorded $\omega \in \Omega$ is real functions $\xi(\omega, \cdot) = \xi(\omega, t)$ of $t \in T$ is called sample distribution function or the trajectory of random function $\xi = \xi(t)$.

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Let us consider certain space X of the real functions x = x (t) of $t \in T$, in which enter all trajectories $\xi = \xi(\omega, t)$, $t \in T$, random function $\xi = \xi(t)$ (for example, this property possesses space $X = R^T$ all real functions x = x (t), $t \in T$). Let us designate $\mathfrak{B}$ minimum σ-algebra of the sets of function space X, which contains all cylindrical multitudes of this space, i.e., the set of the form

$$[x(t_1), \ldots, x(t_n)] \in \Gamma \qquad (2.1)$$

(the set indicated (2.1) it consists of those functions x = x (t), for which the values $[x(t_1), \ldots, x(t_n)]$ in the undertaken points $t_1, \ldots, t_n \in T$ assign the vector, which belongs to borel set Γ N-dimensional vector space R^n). The representation $\xi = \xi(\omega)$, with which each $\omega \in \Omega$ answers the corresponding sample distribution function $\xi(\omega, \cdot) = \xi(\omega, t)$ of $t \in T$ - the cell/element of space X, it is the measurable representation from probability space $(\Omega, \mathfrak{A}, P)$ into the measurable space $(X, \mathfrak{B})$. Sets $A \in \mathfrak{A}$ form $A = \{\xi \in B\}$ - the prototypes of sets $B \in \mathfrak{B}$ with the representation indicated $\xi = \xi(\omega, \cdot)$ - in set is formed σ-algebra. This σ-algebra $\mathfrak{A}_\xi$ is minimum among σ-algebra of the sets, which contain all multitudes of the form

$$[\xi(t_1), \ldots, \xi(t_n)] \in \Gamma \qquad (2.2)$$

(the set indicated consists of those cell/elements $\omega \in \Omega$, for which

value $[\xi(\omega, t_1), \ldots, \xi(\omega, t_n)]$ assign the **vector**, which belongs to borel set Γ n**-dimensional vector** space R^n), **or as otherwise** they speak, σ**-algebra** $\mathfrak{A}_\xi$ **it is generated by values ξ (t),** $t \in T$. The **probability measure** P_ξ, **determined** on σ**-algebra** $\mathfrak{B}$ **by** the relationship/ratio

$$P_\xi(B) = P\{\xi \in B\}, \quad B \in \mathfrak{B}, \qquad (2.3)$$

is called the probability distribution of random function $\xi = \xi$ (t) (in the appropriate function space X).

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Let us turn to the question concerning that, when the **assigned** family of real values ξ (t) = ξ (ω, t) on space Ω (para**meter t passes** set T) it is random function with the assigned probabil**ity** P_ξ, distribution P more precise, when exists the **probability measure** in space Ω, connected with specified **distribution** P_ξ **by** relationship/ratio (2.3). In this case it is **assumed that the set ξ** **(Ω)** all sample distribution functions ξ (ω, $\cdot$) = ξ (ω, t) **of** $t \in T$ **belongs** to space **X**.

It is easy to see that this probability measure P exists when and only when set ξ (Ω) has full/total/complete external measure, i.e.,

$$P_\xi(B)=1 \text{ with } B \supseteq \xi(\Omega) \qquad (2.4)$$

for any measurable multitude $B \in X$.

Actually, if P_ξ is a probability distribution of random function $\xi = \xi(t)$, then for any multitude $\overline{B} \in X$, which lies at addition to set $\xi(\Omega)$, the prototype $\{\xi \in \overline{B}\}$ is void set and

$$P_\xi(\overline{B}) = P\{\xi \in \overline{B}\} = 0.$$

On the other hand, for any multitudes $B_1, B_2 \in \mathfrak{B}$, which have one and the same prototype: $\{\xi \in B_1\} = \{\xi \in B_2\}$, symmetrical difference $B_1 \circ B_2 = (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$ it enters in addition to set $\xi(\Omega)$, and under condition (2.4)

$$P_\xi(B_1 \circ B_2) = 0, \quad P_\xi(B_1) = P_\xi(B_2).$$

Therefore the relationship/ratio

$$P\{\xi \in B\} = P_\xi(B), \quad B \in \mathfrak{B}, \qquad (2.5)$$

determines single-valued function $P = P(A)$ in σ-algebra $\mathfrak{A}_\xi$ all multitudes of form $A = \{\xi \in B\}$, $B \in \mathfrak{B}$. generated values $\xi(t)$, $t \in T$. It is obvious, P to eat a probability measure and $\xi = \xi(t)$ - random function on probability space $(\Omega, \mathfrak{A}, P)$ with the assigned probability distribution P_ξ.

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The probability measure P in σ-algebra $\mathfrak{A}_{\xi}$, generated values ξ (t), $\xi(t)$, $t\in T$, unambiguously is defined by finite-dimensional distributions $\mathsf{P}_{t_1,\ldots,t_n}$, each of which is the Borel measure in the appropriate N-dimensional vector space R^n, defined as

$$\mathsf{P}_{t_1,\ldots,t_n}(\Gamma)=\mathsf{P}\{[\xi(t_1),\ \ldots,\ \xi(t_n)]\in\Gamma\} \qquad (2.6)$$

$\mathsf{P}_{t_1,\ldots,t_n}$ is the probability distribution of random vector $[\xi(t_1),\ \ldots,\ \xi(t_n)]$.

Specifically,

$$\mathsf{P}(A)=\inf\sum_k \mathsf{P}(A_k), \qquad (2.7)$$

where lower bound is taken on all multitudes A_k form (2.2), in set that which cover set $A\in\mathfrak{A}$. Specifically, this is related also to probability distribution P_{ξ} in the appropriate function space X - to probability measure in σ-algebra $\mathfrak{B}$, generated by the directly assigned magnitudes ξ (t) = ξ (x, t) on space X, i.e., by the values of the form

$$\xi(t,\ x)=x(t),\quad x\in X \qquad (2.8)$$

(where fixed/recorded for each functional ξ (x, t) = x (t) of $x\in X$ parameter t passes set T).

Let us designate $\Gamma \times R^{n-m}$ borel set in the N-dimensional space of vectors $[x(t_1), \ldots, x(t_n)]$ such, what $[x(t_{i_1}), \ldots, x(t_{i_m})] \in \Gamma$ (Γ - borel set in m-dimensional subspace $R^m \subseteq R^n$), and remaining coordinates $x(t_i)$ are arbitrary. The finite-dimensional distributions are matched in the sense that

$$P_{t_1, \ldots, t_n}(\Gamma \times R^{n-m}) = P_{t_{i_1}, \ldots, t_{i_m}}(\Gamma) \qquad (2.9)$$

for all multitudes of the type indicated.

Let $X = R^T$ be space of all real functions x = x (t), $t \in T$. According to Kolmogorov's known theorem [1] any matched family of distributions $P_{t_1, \ldots, t_n}$ assigns on the algebra of all cylindrical multitudes (2.1) continuous additive function P (determined by the formula (2.6), in which figure the directly assigned magnitudes of form (2.8)).

FOOTNOTE [1]. See [8], page 150. ENDFOOTNOTE.

This function unambiguously is continued into probability measure in σ-algebra $\mathfrak{B}$. The directly assigned random function $\xi = \xi$ (t) with the values ξ (t indicated) = ξ (x, t) on probability space $(X, \mathfrak{B}, P)$ has the finite-dimensional distributions, which coincide with the initial matched distributions $P_{t_1, \ldots, t_n}$.

Being transmitted of probability distribution $P = P_\xi$ in function space X, under condition (2.4) it is possible to determine (see formula (2.5)) the probabilistic faith in the appropriate space Ω.

Random functions $\xi = \xi$ (t) and $\xi = \xi$ (t) with values on just one probability space are called equivalent, if with probability 1 (for almost all $\omega \in \Omega$)

$$\xi(\omega, t) = \xi(\omega, t)$$

with each fix/recorded $t \in T$. Are obvious, finite-dimensional are obvious, the finite-dimensional distributions of equivalent random functions coincide. By passing to equivalent random function $\xi = \xi$ (t) with trajectories in this or another function space X, it is possible to determine (see formula (2.3)) probability measure, also, in this space.

Random variables are called Gaussian, if Gaussian are their

finite-dimensional distributions. Is more precise (when with certain parametrization we are dealing with random function $\xi = \xi$ (t) of parameter $t \in T$), value ξ (t) = ξ (ω, t) and function itself $\xi = \xi$ (t) are called Gaussian, if Gaussian are all the finite-dimensional distributions $P_{t_1, \ldots, t_n}$. Gaussian is called probability measure P in σ-algebra $\mathfrak{A}_\xi$, generated all values ξ (t).

Each of the finite-dimensional distributions $P_{t_1, \ldots, t_n}$ Gaussian random functions $\xi = \xi$ (t) has the average value $[a(t_1), \ldots, a(t_n)]$ and correlation matrix/die $\{B(t_k, t_l)\}$, where a (t), $t \in T$, - average value of random function $\xi = \xi$ (t), and B (s, t), s, $t \in T$, it is correlation functions [1]:

$$a(t) = M\xi(t),$$
$$B(s, t) = M[\xi(s) - a(s)][\xi(t) - a(t)], \quad s, t \in T. \quad (2.10)$$

FOOTNOTE [1]. $M\xi$ indicates the mathematical expectation of random variable $\xi = \xi$ (ω) on probability space $(\Omega, \mathfrak{A}, P)$: $M\xi = \int_\Omega \xi(\omega) P(d\omega)$. ENDFOOTNOTE.

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Thus, the Gaussian measure P in σ-algebra $\mathfrak{A}_\xi$ unambiguously is

determined by its by the average value of a (t), $t \in T$, and by the correlation function B (s, t), s, $t \in T$.

The average value a (t), $t \in T$, can be arbitrary, and the correlation function B (s, t), s, $t \in T$, satisfies only the condition of the positive definition:

$$\sum_{k,\, l=1}^{n} c_k c_l B(t_k, t_l) \geqslant 0 \qquad (2.11)$$

for any $t_1, \ldots, t_n \in T$ and real $c_1, \ldots, c_n$.

For any functions a (t), $t \in T$, and by positively determined B (s, t), s, $t \in T$, there is Gaussian random function with the average value of a (t), $t \in T$, and by the correlation function B (s, t), s, $t \in T$. Specifically, the Gaussian distributions $P_{t_1, \ldots, t_n}$ with the average values $[a(t_1), \ldots, a(t_n)]$ and correlation matrix/dies $\{B(t_k, t_l)\}$ are matched and is assigned Gaussian measure P in space $X = R^T$ all real functions x = x (t) of $t \in T$, determined in σ-algebra $\mathfrak{B} = \mathfrak{A}_\xi$, which is generated by the directly assigned magnitudes ξ (t) = ξ (x, t) on X of form (2.8) (parameter t passes set T).

§3. Some lemmas about the convergence of Gaussian values.

Let, $\xi_n = \xi_n(\omega)$, n = 1, 2, ..., certain sequence of random variables on probability space $(\Omega, \mathfrak{A}, P)$.

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Sequence ξ_n, n = 1, 2, ..., it is called that which is converging on probability on set $A \in \mathfrak{A}$ to certain value $\xi = \xi(\omega)$, if for any $\varepsilon > 0$

$$\lim_{n \to \infty} P(\{|\xi_n - \xi| > \varepsilon\} \cap A) = 0. \quad (3.1)$$

Recall that the sequence ξ_n, n = 1, 2, ..., descends on probability when and only when it is fundamental [1], i.e., on the same multitude A descends on probability to 0 sequence $\Delta_{nm} = \xi_n - \xi_m$, n, m = 1, 2,: .

FOOTNOTE [1]. See, for example, [8], page 90. ENDFOOTNOTE.

Lemma 1. If the sequence of Gaussian values ξ_n, n = 1, 2, ..., descend on probability on certain multitude $A \in \mathfrak{A}$ positive measure $(P(A) > 0)$, then it it converge in mean:

$$\lim_{n\to\infty} \mathsf{M}\,[\xi_n - \xi]^2 = 0. \qquad (3.2)$$

Proof. Let us consider Gaussian values $\Delta_{nm} = \xi_n - \xi_m$. For any $\varepsilon > 0$

$$\mathsf{P}\{|\Delta_{nm}| > \varepsilon\} = 2\int_{\varepsilon}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_{nm}} \exp\left\{-\frac{(x - a_{nm})^2}{2\sigma_{nm}^2}\right\} dx,$$

where $a_{nm} = \mathsf{M}\Delta_{nm}$, $\sigma_{nm}^2 = \mathsf{M}(\Delta_{nm} - a_{nm})^2$. Let us assume that the sequence ξ_n, n = 1, 2, ..., not converge in mean, which is equivalent to the condition

$$\overline{\lim_{n,\, m\to\infty}}\, (a_{nm}^2 + \sigma_{nm}^2) > 0.$$

It is easy to see that under this condition for certain positive ε

$$\overline{\lim_{n,\, m\to\infty}}\, \mathsf{P}\{|\Delta_{nm}| > \varepsilon\} \geqslant 1 - p/2,$$

where $p = \mathsf{P}(A) > 0$. But then

$$\overline{\lim_{n,\, m\to\infty}}\, \mathsf{P}(\{|\Delta_{nm}| > \varepsilon\} \cap A) \geqslant p/2,$$

which contradicts condition (3.1).

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Therefore

$$\lim_{n,\, m\to\infty} \mathsf{M}\Delta_{nm}^2 = \lim_{n,\, m\to\infty} (a_{nm}^2 + \sigma_{nm}^2) = 0,$$

i.e. sequence ξ_n, n = 1, 2, ..., is fundamental on the average and, therefore, it converge in mean, Q. E. D.

Specifically, if the sequence of Gaussian values ξ_n, n = 1, 2, ..., descend with positive probability (i.e. descends for all ω from certain multitude $A \in \mathfrak{A}$ positive measure), then it it converge in mean.

Let us consider the sequence of independent Gaussian values ξ_n, n = 1, 2,...

Lemma 2. A series $\sum_{n=1}^{\infty} \xi_n^2$ descends with positive probability when and only when descends a series $\sum_{n=1}^{\infty} M\xi_n^2$.

Proof. It is obvious,

$$\sum_{n=1}^{\infty} M\xi_n^2 = M \sum_{n=1}^{\infty} \xi_n^2,$$

and therefore from convergence of series $\sum_{n=1}^{\infty} M\xi_n^2$ it follows that value $\xi^2 = \sum_{n=1}^{\infty} \xi_n^2(\omega)$ is final for almost all $\omega \in \Omega$, i.e., a series descends with probability 1. Let now a series $\sum_{n=1}^{\infty} \xi_n^2$ descend with positive probability (on the strength of known law zero or of one [1] it descends with probability 1).

FOOTNOTE [1]. See, for example, [8], page 157. ENDFOOTNOTE.

Then sequence ξ_n, n = 1, 2, ..., converge to 0 on the average: $M\xi_n^2 \to 0$ with n $\longrightarrow \infty$ (see lemma 1).

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Let us assume $a_n = M\xi_n$, $\sigma_n^2 = M(\xi_n - a_n)^2$. Then

$$a_n^2 + \sigma_n^2 = M(\xi_n')^2 + \int_{|x|>1} x^2 \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left\{-\frac{(x-a_n)^2}{2\sigma_n^2}\right\} dx,$$

where random variables $\xi_n' = \xi_n'(\omega)$ are defined as

$$\xi_n'(\omega) = \begin{cases} \xi_n(\omega) & \text{with } |\xi_n| \leqslant 1, \\ 0 & \text{with } |\xi_n| > 1. \end{cases}$$

With $a_n^2 + \sigma_n^2 \to 0$ we have

$$\int_{|x|>1} x^2 \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left\{-\frac{(x-a_n)^2}{2\sigma_n^2}\right\} dx = o(a_n^2 + \sigma_n^2),$$

so that

$$M(\xi_n')^2 \sim a_n^2 + \sigma_n^2.$$

According to the well known theorem about three series [1] for the convergence (with probability 1) of a series $\sum_{n=1}^{\infty} \xi_n^2$ from independent variables ξ_n^2, n = 1, 2, ..., it is necessary that $\sum_{n=1}^{\infty} M(\xi_n')^2 < \infty$.

FOOTNOTE [1]. See, for example, [8], page 166. ENDFOOTNOTE.

But $M(\xi_n')^2 \sim a_n^2 + \sigma_n^2$, and therefore from convergence of series $\sum_{n=1}^{\infty} \xi_n^2$ follows that $\sum_{n=1}^{\infty} (a_n^2 + \sigma_n^2) < \infty$.

§4. Gaussian values in Hilbert space.

Random variable ξ in Euclidian n-dimensional space R^n is called Gaussian, if Gaussian is its probability distribution.

The random variable $\xi \in R^n$ is Gaussian when and only when with each $u \in R^n$ Gaussian is the real value ξ (u) = (u, ξ) (equal to the scalar product of cell/elements u, $\xi \in R^n$).

Actually, value at point $u \in R^n$ characteristic function ϕ (u) random variable $\xi \in R^n$ coincides with the value of the characteristic function of real random variable ξ (u) = (u, ξ) at point 1 and takes the form

$$\varphi(u) = Me^{i(u, \xi)} = \exp\left\{i(u, a) - \frac{1}{2}(Bu, u)\right\}, \quad u \in R^n$$

(see formula (1.1), where (u, a) - the average value, a (Bu, u) - the dispersion of Gaussian value ξ (u) = (u, ξ)).

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It is obvious that the random variable $\xi \in R^n$ is Gaussian when and only when Gaussian is the random function of form ξ (u) = (u, ξ) of $u \in R^n$.

Let U be the total separable Hilbert space and ξ = ξ (ω) - function on probability space $(\Omega, \mathfrak{A}, P)$ with values in U. Random element ξ Hilbert space U it is called random variable in U, if the scalar product (u, ξ) with each $u \in U$ is real random variable, i.e., a measurable function in probability space $(\Omega, \mathfrak{A}, P)$.

Random variable ξ in Hilbert space U is called Gaussian, if with each $u \in U$ Gaussian is the real random variable ξ (u) = (u, ξ). This is equivalent to the fact that Gaussian is the random function ξ (u) = (u, ξ) of $u \in U$, since Gaussian will be not only the separate values ξ (u) = (u, ξ), but also any vector values $[\xi(u_1), \ldots, \xi(u_n)]$.

In fact, for any vector $\lambda = [\lambda_1, \ldots, \lambda_n]$ in N-dimensional vector space R^n the scalar product $\sum_{k=1}^{n} \lambda_k \xi(u_k)$ is

$$\sum_{k=1}^{n} \lambda_k \xi(u_k) = \left(\sum_{k=1}^{n} \lambda_k u_k, \xi\right) = (u, \xi),$$

where $u = \sum_{k=1}^{n} \lambda_k u_k \in U$, and according to condition value $\xi(u) = (u, \xi)$ is Gaussian.

Is obvious, the average value

$$a(u) = \mathsf{M}(u, \xi), \quad u \in U,$$

random function $\xi(u) = (u, \xi)$, $u \in U$, is linear functional, and the correlation function

$$B(u, v) = \mathsf{M}[(u, \xi) - a(u)][(v, \xi) - a(v)], \quad u, v \in U,$$

- by bilinear positive functional on hilbert space U.

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In this case, since with each fix/recorded $\omega \in \Omega$ the scalar product (u, ξ) is a continuous function of $u \in U$, for Gaussian function $\xi(u) = (u, \xi)$ of $u \in U$ it must occur and continuity on the average (see lemma 1):

$$\lim_{\|u-v\| \to 0} \mathsf{M}[(u, \xi) - (v, \xi)]^2 = 0 \qquad (4.1)$$

($\|u\|$ indicates the norm of cell/element $u \in U$). But

$$M[(u, \xi) - (v, \xi)]^2 = a(u-v)^2 + B(u-v, u-v)$$

and condition (4.1) indicates the continuity of functionals a (u) and B (u, v).

As any linear continuous functional, the average value a (u) is representable in the form

$$a(u) = (u, a), \quad u \in U. \qquad (4.2)$$

The cell/element $a \in U$, indicated which possesses those by property, which

$$(u, a) = \int_{\Omega} (u, \xi(\omega))\, P(d\omega) \qquad (4.3)$$

with all $u \in U$, is called the average value [1] random variable $\xi \in U$.

FOOTNOTE [1]. Apropos of the integrability of functions with values in Hilbert space see, for example, [26], page 59. ENDFOOTNOTE.

As any bilinear positive continuous functional, the correlation function B (u, v) is representable in the form

$$B(u, v) = (Bu, v), \quad u, v \in U, \qquad (4.4)$$

where B is linear positive correlation by operator in Gilpert space U, called the correlation operator.

Let us show that the correlation operator B is completely continuous.

Actually, any orthonormalized sequence v_1, v_2 ... weakly converge to 0, so that at all $\omega \in \Omega$ Gaussian values $\xi_n = (v_n, \xi)$, n = 1, 2, ..., where $\xi = \xi(\omega) \in U$, converge to 0 with n $\rightarrow \infty$.

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Consequently (see lemma 1), they descend on the average, i.e.,

$$M\xi_n^2 = (Bv_n, v_n) \rightarrow 0$$

(here and throughout we consider for simplicity of notation that the average value $a \in U$ is equal to 0). If one assumes that the operator B is not completely continuous, then outside certain vicinity of zero it found the infinite number of points of the spectrum (taking into account multiplicity) and, therefore, the infinite number of invariant orthogonal subspaces, for each of cell/elements of which

$$(Bu, u) = \int_{|\lambda| > \varepsilon} \lambda \, d(E_\lambda u, u) \geqslant \varepsilon \| u \|^2,$$

where $B = \int \lambda \, dE_\lambda$ - the spectral representation of the continuous symmetrical operator B.

Further, let us select the full/total/complete orthonormalized basis from its own cell/elements $v_1, v_2 \ldots$ this completely continuous symmetrical positive operator B, which correspond to the eigenvalues $\sigma^2_1, \sigma^2_2, \ldots$. The corresponding values $\xi_k = (v_k, \xi)$, $k = 1, 2, \ldots$, are not correlated:

$$M\xi_k\xi_j = (Bv_k, v_j) = \begin{cases} \sigma_k^2 & \text{with } j = k, \\ 0 & \text{with } j \neq k. \end{cases}$$

In this case

$$\sum_1^\infty \xi_k^2(\omega) = \sum_1^\infty (v_k, \xi(\omega))^2 = \| \xi(\omega) \|^2.$$

As is known, the uncorrelated Gaussian values are independent variables, and from convergence of series $\sum_1^\infty \xi_k^2(\omega)$ (with all ω) comes convergence of series $\sum_1^\infty M\xi_k^2$ (see lemma 2).

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Consequently,

$$\sum_1^\infty (Bv_k, v_k) = \sum_1^\infty M\xi_k^2 = \sum_1^\infty \sigma_k^2 < \infty,$$

i.e. the correlation operator B is nuclear [1]: for any orthonormal set $u_1, u_2, \ldots \in U$

$$\sum_1^\infty (Bu_k, u_k) < \infty. \qquad (4.5)$$

FOOTNOTE [1]. See, for example, [5], page 55. ENDFOOTNOTE.

Thus, if there is a Gaussian random variable $\xi \in U$, then random function ξ (u) = (u, ξ) the parameter $u \in U$ has the average value of form (4.2) and the correlation function of form (4.4), where the correlation operator B is nuclear operator in hilbert space U.

Let, now ξ (u), $u \in U$, be an arbitrary Gaussian random function with the average value of form (4.2) and by the correlation function of form (4.4), where B is a nuclear operator in hilbert space U. Then there are equivalent by it random function ξ (u), $u \in U$, and Gaussian random variable $\xi = \xi$ (ω) in U such, that

$$\xi(u) = (u, \xi), \quad u \in U. \qquad (4.6)$$

Value $\xi \in U$ indicated at almost all elementary issues ω can be

determined by the formula

$$\xi(\omega)=\sum_{k=1}^{\infty}\xi(v_k)\,v_k, \qquad (4.7)$$

where v_1, v_2 ... - the full/total/complete orthonormal set from nuclear operator's their own cell/elements B, and on the strength of the relationship/ratio

$$M\sum_{k=1}^{\infty}\xi(v_k)^2=\sum_{k=1}^{\infty}B(v_k,v_k)<\infty$$

for independent Gaussian values $\xi(v_1)$, $\xi(v_2)$, ... a series $\sum_{k=1}^{\infty}\xi(v_k)^2$ descends with probability 1.

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In fact, $\xi(u)$, $u\in U$, is random linear functional in the sense that with the probability of 1

$$\xi(\lambda_1u_1+\lambda_2u_2)=\lambda_1\xi(u_1)+\lambda_2\xi(u_2)$$

where any real λ_1, λ_2 and any cell/elements u_1, $u_2\in U$, since as is easy to check,

$$M[\xi(\lambda_1u_1+\lambda_2u_2)-\lambda_1\xi(u_1)-\lambda_2\xi(u_2)]^2=0.$$

Furthermore, this random functional $\xi(u)$ is continuous on the average (see (4.1) and further), and since

$$u = \lim_{n\to\infty} \sum_{k=1}^{n} (u, v_k) v_k,$$

that

$$\xi(u) = \lim_{n\to\infty} \xi\left(\sum_{k=1}^{n} (u, v) v_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} (u, v_k) \xi(v_k)$$

(in the sense of mean convergence), but at the same time with probability 1

$$(u, \xi) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(u, \sum_{k=1}^{n} \xi(v_k) v_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} (u, v_k) \xi(v_k),$$

so that with probability 1 for each value ξ (u) the initial random function ξ (u), $u \in U$, it occurs equality (4.6).

Thus, we arrived at the following result [1].

FOOTNOTE [1]. Survey/coverage of results about distributions in linear spaces is available, for example, from ~~S.~~ Yu. V. Prokhorov (Yu. V Prohorov, The method of characteristic functionals, Proc. 4th Berkeley sympos., Vol. 2, 1961, 403-419). ENDFOOTNOTE.

Theorem 1. Gaussian functional ξ (u), $u \in U$, on Hilbert space U let us present in formula (4.6) when and only when the average value a (u), $u \in U$, is linear, and the correlation function B (u, v), u,

$v \in U$, bilinear continuous functionals, whereupon in representation (4.4) of the bilinear functional B (u, v) the corresponding operator B is nuclear.

As is known, hilbert space U it is possible to identify with the conjugated/combined space X of all linear continuous functionals on U.

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Specifically, each such functional $x \in X$ unambiguously is assigned by formula x = (u, x), $u \in U$, where $x \in U$ - the determined cell/element hilbert space U. According to that which was presented in §2 any Gaussian random function ξ (u), $u \in U$, form (4.6) corresponds the probability distribution P_ξ in hilbert space X = U ("selective functionals" ξ (u) = (u, ξ (ω)), $u \in U$, they belong to X). Gaussian measure P_ξ is determined in σ-algebra $\mathfrak{B}$, generated by cylindrical multitudes of space X = U of the form

$$[(u_1, x), \ldots, (u_n, x)] \in \Gamma, \qquad (4.8)$$

where $u_1, \ldots, u_n \in U$ and Γ - the Borel sets of n-dimensional Euclidian space; σ-algebra $\mathfrak{B}$ indicated is generated, obviously, by the directly assigned magnitudes of the form

$$\xi(x, u) = (u, x), \quad x \in X, \qquad (4.9)$$

where parameter u passes space U = X.

For any cell/element $a \in U$ and the positive nuclear operator B there is Gaussian random function ξ (u), $u \in U$, form (4.6) with the average value $a \in U$ and the correlation operator B. There is directly assigned Gaussian function of such type with the values, determined by formula (4.9) on probability space $(X, \mathfrak{B}, P_\xi)$.

Example. Gaussian values in function space $\mathscr{L}^2(T)$.

Let ξ = ξ (t) be a Gaussian random process on segment T of real straight line with the average value of a (t), $t \in T$, and by the correlation function B (s, t), s, $t \in T$, which they satisfy the condition: with all s, $t \in T$

$$\lim_{s \to t} [a(s) - a(t)] = 0, \qquad (4.10)$$
$$\lim_{s \to t} [B(s, s) - 2B(s, t) + B(t, t)] = 0.$$

This condition indicates continuity on the average of random process ξ = ξ (t):

$$\lim_{s \to t} M [\xi(s) - \xi(t)]^2 = 0.$$

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As is known [1], in this case there is an equivalent measurable process (with values $\bar{\xi}(t) = \bar{\xi}(\omega, t)$), i.e., such, that the function $\bar{\xi} = \bar{\xi}(\omega, t)$ the pair of variables (ω, t) on product space $\Omega \times T$ is measurable.

FOOTNOTE [1]. See, for example, [8], page 209. ENDFOOTNOTE.

Let us consider that measurable is initial Gaussian process itself $\xi = \xi(t)$. Let us assume that is satisfied also the condition:

$$\int_T a^2(t)\,dt < \infty, \quad \int_T B(t, t)\,dt < \infty. \qquad (4.11)$$

This condition means that

$$\int_T M\xi^2(t)\,dt < \infty.$$

According to Fubini's theorem about the repeated integration

$$\int_T M\xi^2(t)\,dt = \int_\Omega \int_T \xi^2(\omega, t)\,dt\,P(d\omega) < \infty$$

and almost all the sample distribution functions $\xi(\omega, \cdot) = \xi(\omega, t)$ of $t \in T$ belong to hilbert space $\mathscr{L}^2(T)$ the real integrated squared functions $u = u(t)$ of $t \in T$ with the scalar product

$$(u, v) = \int_T u(t) v(t) dt.$$

Redefining the value ξ (ω, t) for those $\omega \in \Omega$, with which sample distribution functions ξ $(\omega, \cdot) = \xi$ (ω, t), $t \in T$, not enters in $\mathcal{L}^2$ (many such $\omega \in \Omega$ have measure as 0), it is possible to pass to the measurable Gaussian process $\xi = \xi$ (t), all the sample distribution functions of which belong to hilbert space $\mathcal{L}^2$. This random function $\xi =$ $= \xi$ $(\omega, \cdot)$ can be considered as random element in hilbert space $\mathcal{L}^2$.

Let us consider the scalar products

$$(u, \xi) = \int_T u(t) \xi(\omega, t) dt, \quad u \in U.$$

end section.

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Since $\xi = \xi(\omega, t)$ there is the measurable function of variables (ω, t), with each fix/recorded $u \in U$ real function (u, ξ) of $\omega \in \Omega$ also is measureable - it is random variable. Thus, $\xi = \xi(\omega, \bullet)$ - random variable in hilbert space $\mathscr{L}^2(T)$. According to Fubini's theorem the random function $\xi(u) = (u, \xi)$ of $u \in U$ has the average value

$$a(u)=\int_T u(t)\,a(t)\,dt=(u,\ a),\quad u\in U,\qquad (4.12)$$

and the correlation function

$$B(u,\ v)=\int_T\int_T u(s)\,v(t)\,B(s,\ t)\,ds\,dt=(Bu,\ v),\ u,\ v\in U,\ (4.13)$$

where the correlation operator B it is assigned by nucleus B (s, t):

$$Bu(t)=\int_T B(s,\ t)\,u(s)\,ds.$$

The random variable $\xi \in \mathscr{L}^2$ is Gaussian. Actually, as can easily be seen, for the continuous function u = u (t) of $t \in T$ random variable $(u, \xi) = \int_T u(t)\,\xi(t)\,dt$ is limit on the average of the Gaussian values of the form

$$\sum_{k=1}^{n} u(t_k)\,\xi(t_k)\,(t_k - t_{k-1}),$$

where $t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n$ - some points of the separation of segment T,

and for arbitrary function $u \in \mathscr{L}^2$ value (u, ξ) is limit on the average of Gaussian values (u_n, ξ), where $u_n = u_n(t)$, n = 1, 2, ..., the sequence of continuous functions, which converges on the average to function u = u (t). The maximum value for the sequence of Gaussian values is, as is known, also Gaussian [1].

FOOTNOTE [1]. See, for example [8], page 33. ENDFOOTNOTE.

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§5. Conditional probability distributions and conditional mathematical expectations.

Let ξ (u) = ξ (ω, u), $u \in U$, are a family of Gaussian random variables on probability space Ω and $\mathfrak{A}(U)$ is σ-algebra of the sets of space Ω, generated by all values ξ (u) = ξ (ω, u) on Ω (parameter u passes certain multitude U). For simplicity let us consider that $M\xi(u) = 0$, $u \in U$.

Let us designate H (U) the hilbert space of random variables η (measurable relative to σ-algebra $\mathfrak{A}$ (U)) with the scalar product

$$\langle \eta_1, \eta_2 \rangle = M(\eta_1 \cdot \eta_2). \qquad (5.1)$$

Let us designate H (U) the locked linear closure of all values ξ (u), $u \in U$. Let S and T be some subsets in U.

Let us consider arbitrary value $\eta \in H(S)$ and its projection $\hat{\eta}$ on subspace H (T). Since H (U) there is a set Gaussian values, difference $\Delta = \eta - \hat{\eta}$, as the Gaussian value, orthogonal to all values ξ (t), $t \in T$, does not depend on these values. Thus,

$$\eta = \hat{\eta} + \Delta, \qquad (5.2)$$

where $\hat{\eta}$ there is value described above, measurable relative to σ-algebra $\mathfrak{A}(T)$, a Δ - independent variable of ξ (t), $t \in T$, Gaussian value with zero average value and dispersion $\sigma^2 = M(\eta - \hat{\eta})^2$. It is easy to see that the conditional distribution [1] value η relative to σ-algebra $\mathfrak{A}(T)$ always exists and with almost all ω is Gaussian distribution with the average value

$$M\{\eta/\mathfrak{A}(T)\} = \hat{\eta}(\omega) \qquad (5.3)$$

and the constant dispersion

$$\sigma^2 = M\{(\eta - \hat{\eta})^2/\mathfrak{A}(T)\} = M(\eta - \hat{\eta})^2. \qquad (5.4)$$

FOOTNOTE [1]. See, for example, [12], page 75. ENDFOOTNOTE.

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Recall that conditional mathematical expectation $M\{\eta/\mathfrak{A}(T)\}$ geometrically is the projection of value $\eta \in H(U)$ on subspace $H(T)$ and in this case this is value $\hat{\eta}$ is a projection on subspace H (T).

As is known, for Gaussian values there are all torque/moments. Let us designate H^n (U) the closing/shorting of the subspace of all values of the form

$$\eta = \varphi[\xi(u_1), \ldots, \xi(u_k)], \tag{5.5}$$

where ϕ (x_1, ..., x_k) - the polynomial of degree not higher than n from the arbitrary number of variables x_1, ..., x_k, and u_1, ..., $u_k \in U$.

Theorem 2. For any value $\eta \in H^n$ (S) its conditional mathematical expectation $\hat{\eta} = M\{\eta/\mathfrak{A}(T)\}$ enters in subspace $H^n(T)$ (with those index n):

$$M\{\eta/\mathfrak{A}(T)\} \in H^n(T). \tag{5.6}$$

Proof. Without limiting generality, it is possible to count that sets S and T are final (let us say, S = {s_1, ..., s_k} and T = {t_1, ..., t_k}). In fact, to the general case it is possible to pass by passage to the limit [1] since

$$\eta = \lim_{m\to\infty} \eta_m, \quad M\{\eta/\mathfrak{A}(T)\} = \lim_{m\to\infty} M\{\eta_m/\mathfrak{A}(T)\},$$

where there is in form a mean convergence and

$$\eta_m = \varphi_m[\xi(s_{k_1}), \ldots, \xi(s_{k_m})],$$

and also

$$M[\eta/\mathfrak{A}(T)] = \lim_{m\to\infty} M[\eta/\xi(t_{k_1}), \ldots, \xi(t_{k_m})].$$

FOOTNOTE [1]. See, for example, [12], page 29, 287. ENDFOOTNOTE.

Let $\eta=\varphi[\xi(s_1), \ldots, \xi(s_k)]$, where ϕ (x_1, ..., x_k) - the polynomial of degree be not higher than n. As has already been indicated, theorem was accurate for n = 1. Let us assume that it is accurate for all indices, which do not exceed n - 1. Let us designate $\hat{\xi}(s_j)$ the projections of values $\xi(s_j)$, j = 1, ..., k, on subspace H (T). Differences $\xi(s_j)-\hat{\xi}(s_j)$, j = 1, ..., k, do not depend on values ξ (t), $t\in T$.

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Let us assume

$$\zeta=\varphi[\xi(s_1)-\hat{\xi}(s_1), \ldots, \xi(s_k)-\hat{\xi}(s_k)].$$

Value ζ does not depend on ξ (t), $t\in T$, and in the expansion

$$\eta-\zeta=\sum_{j=1}^{k}\frac{\partial}{\partial x_j}\varphi[\xi(s_1), \ldots, \xi(s_k)]\hat{\xi}(s_j)+\ldots$$

the right side is the linear combination of the expressions of the form

$$\varphi_m[\xi(s_1), \ldots, \xi(s_k)] \cdot \psi_{n-m}[\hat{\xi}(s_1), \ldots, \hat{\xi}(s_k)],$$

where $\varphi_m(x_1, \ldots, x_k)$ and $\psi_{n-m}(x_1, \ldots, x_k)$ - the polynomials of degree not higher than m and n - m respectively, whereupon $m \leqslant n - 1$. By hypothesis conditional mathematical expectation $\hat{\eta}_m = \mathrm{M}[\eta_m/\mathfrak{A}(T)]$ value $\eta_m = \varphi_m[\xi(s_1), \ldots, \xi(s_k)]$ enters in subspace $H^m(T)$, $m \leqslant n - 1$. It is obvious, product $\hat{\eta}_m \cdot \psi_{n-m}[\hat{\xi}(s_1), \ldots, \hat{\xi}(s_k)]$ enters in subspace $H^n(T)$. Enters in $H^n(T)$ and the linear combination of such products, which is the conditional mathematical expectation of difference $\eta - \zeta$, equal to $\hat{\eta} - \mathrm{M}\zeta$. Thus, $\hat{\eta}_m = \mathrm{M}[\eta/\mathfrak{A}(T)] \in H^n(T)$,

Q. E. D.

Let us determine Hermite's polynomials of many variables.

Let $P(dx)$ be a Gaussian measure in k-dimensional space R^k vectors $x = [x_1, \ldots, x_k]$ and H - the hilbert space of all real integrated squared functions $\phi = \phi(x)$ of $x \in R^k$ with the scalar product

$$\langle \varphi, \psi \rangle = \int_{R^k} \varphi(x)\psi(x)P(dx).$$

As is known [1] Gaussian measure has all torque/moments, whereupon the set of all polynomials $\phi = \phi(x_1, \ldots, x_k)$ of the variables $x_1, \ldots, x_k$ is a everywhere dense set in H.

FOOTNOTE [1]. This can be clarified as follows. It is obvious, the set of functions of form $e^{i(t,x)}, (t,x) = \sum t_k x_x,$ is full/total/complete in composite space $H,$ and

$$\left\| e^{i(t,x)} - \sum_{n=1}^{N} \frac{[i(t,x)]^n}{n!} \right\|^2 \leqslant C \frac{\sigma^{2(n+1)}}{(n+1)!} \to 0, \text{ where } \sigma^2 = \int (t,x)^2 P(dx).$$

ENDFOOTNOTE.

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Any polynomial $\phi = \phi(x_1, \ldots, x_k)$ degree p, orthogonal to all polynomials of the degree smaller than p, let us call Hermite's polynomial.

Let us designate H_p the set of Hermite's all polynomials one and the same degree p. It is obvious, H_p a finite-dimensional subspace, whereupon hilbert space H is a sum of orthogonal subspaces H_p, p = 0, 1, ...:

$$H = \sum_{p=0}^{\infty} \oplus H_p.$$

Let us consider Gaussian vector quantity $[\xi(u_1), \ldots, \xi(u_1), \ldots, \xi(u_k)]$. Let us designate $H_p(u_1, \ldots, u_k)$ the set of all values of the form

$$\eta = \varphi[\xi(u_1), \ldots, \xi(u_k)],$$

where $\phi = \phi(x_1, \ldots, x_k)$ - the Hermite polynomial of degree p relative to probability distribution P vector quantity $[\xi(u_1), \ldots, \xi(u_k)]$. It is obvious,

$$H(u_1, \ldots, u_k) = \sum_{p=0}^{\infty} \oplus H_p(u_1, \ldots, u_k). \qquad (5.7)$$

Lemma 3. Whatever $s_1, \ldots, s_k$ and $t_1, \ldots, t_l$, for different p and q, subspace $H_p(s_1, \ldots, s_k)$ and $H_q(t_1, \ldots, t_l)$ were orthogonal:

$$H_p(s_1, \ldots, s_k) \perp H_q(t_1, \ldots, t_l) \quad \text{with} \quad p \neq q. \qquad (5.8)$$

¶ Proof. Let for certainty p < q. Let us consider arbitrary value $\eta \in H_p(s_1, \ldots, s_k)$ and its conditional mathematical expectation $\hat{\eta} = M\{\eta/\mathfrak{A}(t_1, \ldots, t_l)\}$. According to theorem 2 value $\hat{\eta}$ enters in subspace $H^p(t_1, \ldots, t_l) = \sum_{r=0}^{p} \oplus H_r(t_1, \ldots, t_l)$. But $\hat{\eta}$ there is a projection of value η on whole subspace $H(t_1, \ldots, t_l)$, so that difference $\eta - \hat{\eta}$ is orthogonal $H(t_1, \ldots, t_l)$ and, in particular, $\eta - \hat{\eta} \perp H_q(t_1, \ldots, t_l)$.

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Simultaneously $\hat{\eta} \perp H_q(t_1, \ldots, t_l)$, since $\hat{\eta}$ enters in subspace $H^p(t_1, \ldots, t_l)$, orthogonal $H_q(t_1, \ldots, t_l)$ with p < q. Consequently

$$\eta = [(\eta - \hat{\eta}) + \hat{\eta}] \perp H_q(t_1, \ldots, t_l),$$

Q. E. D.

Further, let us define subspace $H_p(U)$ as the locked linear closure of all subspaces $H_p(u_1, \ldots, u_k)$, where $u_1, \ldots, u_k \in U$ (but index p one and the same with all $u_1, \ldots, u_k$). It is obvious, on the strength of relationship/ratios (5.7), (5.8) we have

$$H^n(U) = \sum_{p=0}^{n} \oplus H_p(U), \quad H(U) = \sum_{p=0}^{\infty} \oplus H_p(U), \qquad (5.9)$$

where H_0 (U) it contains only constant values; H_1 (U) = H (U) there is the locked linear closure of values ξ (u) $\in U$; H_2 (U) is the locked linear closure of values [ξ (u_1) ξ (u_2) - B (u_1, u_2)], where u_1, u_2, $u_2 \in U$;
H_3 (U) there is the locked linear closure of values [ξ (u_1) ξ (u_2) ξ (u_3) - ξ (u_1) B (u_2, u_3) - ξ (u_2) B (u_1, u_3) - ξ (u_3) B (u_1, u_2)], where u_1, u_2, $u_3 \in U$, and so forth.

As can be seen from expansion (5.9), conditional mathematical expectation $M[\eta/\mathfrak{A}(T)]$ any value $\eta \in H_p(S)$ relative to σ-algebra $\mathfrak{A}(T)$ enters in subspace $H_p(T)$ (with those index p).

Let us note common/general/total formula for the torque/moments:

$$M\xi(u_1) \ldots \xi(u_n) = \sum \prod B(u_k, u_l), \qquad (5.10)$$

where the sum is taken on all separations of set ($u_1, \ldots, u_n$) into vapors (u_k, u_l), andproduct - on all vapors (u_k, u_l) the corresponding separation.

This formula is obtained from the relationship/ratio

$$M\xi(u_1)\ldots\xi(u_n)=\frac{\partial^n}{\partial\lambda_1\ldots\partial\lambda_n}\varphi(0),$$

where $\phi(\lambda) = M\exp\{i(\lambda,\xi)\}$ - the characteristic function of Gaussian vector $\xi = [\xi(u_1), \ldots, \xi(u_n)]$ - it takes the form (see (1.1))

$$\varphi(\lambda)=\exp\left\{-\frac{1}{2}\sum_{k,\,l=1}^{n}\lambda_k\lambda_l B(u_k,\,u_l)\right\},\quad \lambda=[\lambda_1,\,\ldots,\,\lambda_n].$$

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§6. Gaussian stationary processes and spectral representations.

Gaussian random process $\xi(t) = \xi(\omega, t)$ the integral (discrete) or real parameter $t, -\infty < t < \infty$ (with values on probability space Ω), it is called stationary, if its average value constantly

$$a(t)=M\xi(t)\equiv a,$$

and the correlation function B (s, t) depends only on difference s - t:

$$B(s,\,t)=M[\xi(s)-a][\xi(t)-a]=B(s-t) \qquad (6.1)$$

(subsequently let us count a = 0).

Function B (t) in relationship/ratio (6.1) is called the correlation function of stationary process ξ (t); it is representable in the form

$$B(t)=\int e^{i\lambda t}F(d\lambda), \qquad (6.2)$$

where F (dλ) - the so-called spectral measure of stationary process ξ (t) (positive limited measure). Integration in formula (6.2) occurs within limits $-\pi \leqslant \lambda \leqslant \pi$ in the case of discrete "time" of t and within limits $-\infty < \lambda < \infty$ in the case of the continuous time t.

Stationary process itself ξ (t) allow/assumes spectral representation of the form

$$\xi(t)=\int e^{i\lambda t}\Phi(d\lambda), \qquad (6.3)$$

where $\Phi(d\lambda)$ — the so-called stochastic spectral measure such, that

$$M\Phi(\Delta_1)\overline{\Phi(\Delta_2)}=F(\Delta_1\cap\Delta_2).$$

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Any value η from the locked linear closure H (T) of values ξ (t), $\in T$ allow/assumes spectral representation of the form

$$\eta=\int\varphi(\lambda)\Phi(d\lambda), \qquad (6.4)$$

where φ (λ) is a function from space L_T (F) - the real linear closure of functions $e^{i\lambda t}$ of λ, $t\in T$, to that locked according to the scalar product

$$\langle\varphi_1,\varphi_2\rangle_F=\int\varphi_1(\lambda)\overline{\varphi_2(\lambda)}F(d\lambda), \qquad (6.5)$$

whereupon stochastic integral (6.4) is determined for any function $\varphi\in L_T$ (F) and it assigns value $\eta\in H$ (T). Conformity $\eta\leftrightarrow\varphi(\lambda)$ is unitary

isomorphism [1] hilbert spaces H (T) and L_T (F):

FOOTNOTE [1]. Apropos of entire aforesaid see, for example, [22]. ENDFOOTNOTE.

In the case, when parameter t varies continuously and set T is the finite interval, space L_T (F) it would be possible to define as closing/shorting of subspace L^0 (on scalar product (6.5)) all functions of the form

$$\varphi(\lambda)=\int_T e^{i\lambda t}u(t)\,dt, \qquad (6.7)$$

where u = u (t) - the infinitely differentiated functions, which were being converted in 0 outside interval of T. Since with $\lambda \to \infty$ such functions ϕ (λ) they decrease faster than any degree $|\lambda|^{-n}$, that scalar product of type (6.5) can be introduced on subspace L^0 not only with the help of the final spectral measure, but also with the help of any σ-finite measure G ($d\lambda$), that satisfies with certain natural n the condition

$$\int(1+\lambda^2)^{-n}\,G(d\lambda)<\infty.$$

Specifically, let us assume

$$\langle\varphi_1,\varphi_2\rangle_G=\int\varphi_1(\lambda)\overline{\varphi_2(\lambda)}\,G(d\lambda) \qquad (6.8)$$

and let us define full/total/complete hilbert space $L_T(G)$ as closing/shorting of all functions of form (6.7) according to scalar product (6.8).

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Let L_T (G) - be a hilbert space of the type indicated. Formula (6.4) assigns the random functional $\eta = \eta$ (ϕ), determined on the everywhere dense subspace of functions $L_T(G) \cap L_T(F)$. Under which conditions $\eta = \eta(\varphi)$ is (with an accuracy to equivalency) random element from the conjugated/combined space to L_T (G) - by Gaussian linear continuous functional on hilbert space L_T (G)?

In the case, when $\eta = \eta(\varphi)$ is random element from the conjugated/combined space L_T (G), i.e.,

$$\eta(\varphi) = \langle \varphi, \eta \rangle_G, \qquad (6.9)$$

where $\eta = \eta(\lambda)$ – certain Gaussian function with to trajectories from hilbert space L_T (G), the correlation operator B it is possible to determine from the relationship/ratios

$$\begin{aligned} \langle B\varphi_1, \varphi_2 \rangle_G = M\eta(\varphi_1)\eta(\varphi_2) = \langle \varphi_1, \varphi_2 \rangle_F = \\ = \langle A\varphi_1, A\varphi_2 \rangle_F = (A^*A\varphi_1, \varphi_2 \rangle_G, \end{aligned}$$

where A is an operator from hilbert space L_T (G) into hilbert space L_T

(F), determined by the equality

$$A\varphi(\lambda)=\varphi(\lambda), \quad \varphi\in L_T(G)\cap L_T(F), \qquad (6.10)$$

and A* - the adjoint to it operator from L_T (F) in L_T (G). As any correlation operator, B must be nuclear.

On the other hand, if the operator B = A*A is nuclear, then according to theorem 1 Gaussian linear functional $\eta=\eta(\varphi)$ with correlation functional $\langle B\varphi_1,\varphi_2\rangle_G$ is equivalent to certain Gaussian cell/element in the conjugated/combined space to L_T (G), representable by formula (6.9).

Let us note that for not only the nuclear, but also simply bounded operator B = A*A occurs the connection/inclusion

$$L_T(G)\subseteq L_T(F),$$

since

$$\|\varphi\|_F^2=\langle A\varphi,\ A\varphi\rangle_F=\langle B\varphi,\ \varphi\rangle_G\leqslant\|B\|\cdot\|\varphi\|_G^2.$$

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Let us note that for the final measure G (dλ) formula (6.9) is equivalent to the following spectral representation of the initial stationary process ξ (t), $t\in T$:

$$\xi(t)=\int e^{-i\lambda t}\eta(\lambda)\,G(d\lambda), \quad t\in T. \qquad (6.11)$$

In fact, function $\varphi(\lambda)=e^{i\lambda t}$ is formed full/total/complete system in

hilbert space L_T (G) and formula (6.11), which means that $\eta(e^{i\lambda t}) = \langle e^{i\lambda t}, \eta\rangle_G$, $t \in T$, extends to whole space L_T (G) - the locked linear closure of the functions of form $\varphi(\lambda) = e^{i\lambda t}$.

In order to deal directly with the initial random process ξ (t), $t \in T$, and not as functional of it $\eta(\varphi)$, $\varphi \in L_T(G)$, convenient to introduce space X of all real functions x = x (t), $t \in T$, which allow/assume "spectral representation" in the form

$$x(t) = \int e^{-i\lambda t} \psi(\lambda)\, G(d\lambda), \quad t \in T, \qquad (6.12)$$

where $\psi(\lambda) \in L_T(G)$, value x (t) coincide with values of the linear continuous functional $\langle \varphi, \psi \rangle_G$ with φ (λ)=$e^{i\lambda t}$. It is clear that formula (6.12) gives one-to-one conformity between cell/elements $x \in X$ and $\psi \in L_T(G)$.

If we introduce scalar product then

$$\langle x_1, x_2 \rangle = \langle \psi_1, \psi_2 \rangle_G \qquad (6.13)$$

(where ψ_1 and ψ_2 they correspond x_1 and x_2), then X will become full/total/complete hilbert space.

Let us consider separately the case when G (dλ) = 1/2πdλ and hilbert space L_T (G) consists of the functions of the form

$$\psi(\lambda) = \int_T e^{i\lambda t} x(t)\, dt, \qquad (6.14)$$

where x = x (t) it enters in classical space $\mathcal{L}^2(T)$ the real integrated squared functions with the scalar product

$$\langle x_1, x_2\rangle = \int_T x_1(t)\, x_2(t)\, dt,$$

and formula (6.14) it indicates Fourier's transformation.

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From the known equality of Plansherel

$$\langle x_1, x_2\rangle = \langle \psi_1, \psi_2\rangle_G = \int \psi_1(\lambda)\, \overline{\psi_2(\lambda)}\, \frac{d\lambda}{2\pi},$$

so that if we formulas (6.11), (6.12) understand in the sense of Fourier transform, then hilbert space X of all integrated squared functions x = x (t), $t \in T$, formally corresponds to the common diagram of the construction of hilbert spaces with scalar product (6.13).

Above actually we demonstrated the following.

Theorem 3. The random process ξ (t), $t \in T$, is (with an accuracy to equivalency) random element hilbert space X when and only when for that determined by operator's equality (6.10) A the product B = A*A is nuclear operator in hilbert space $L_T(G)$.

Later it will be actually establish/installed that in the case

of the very broad class of absolutely continuous measures $F(d\lambda)$ and $G(d\lambda)$ with densities $f(\lambda) = F(d\lambda)/d\lambda$ and $g(\lambda) = G(d\lambda)/d\lambda$

the operator A^*A is nuclear when

$$\int \frac{f(\lambda)}{g(\lambda)} d\lambda < \infty, \tag{6.15}$$

(whereupon there is also a broad class of the cases, when with to the disturbance of condition (6.15) the operator A^*A is not nuclear).

Specifically, in chapter III is examined the operator of form $\Delta = A^*_1 A_1 - E$, where A^*_1 is an operator of the same type, as A, but reflecting L_T (G) into the space L_T (G_1), constructed with G_1 ($d\lambda$) with density $g_1(\lambda) = g(\lambda) + f(\lambda)$. Since $f(\lambda) = g_1(\lambda) - g(\lambda)$, that

$$\langle \Delta\varphi, \psi\rangle_G = \langle A_1^* A_1 \varphi, \psi\rangle - \langle \varphi, \psi\rangle_G = \\ = \langle \varphi, \psi\rangle_{G_1} - \langle \varphi, \psi\rangle_G = \langle \varphi, \psi\rangle_F = \langle A^* A \varphi, \psi\rangle_G$$

with any ϕ, $\psi \in L_T(G)$, and, therefore, operator A^*A coincides with Δ. It is proved (see theorem 17 chapter III), that the condition

is sufficient in order that the operator $\Delta = A^*A$ would be the Hilbert-Schmidt operator.

$$\int \left[\frac{f(\lambda)}{g(\lambda)}\right]^2 d\lambda < \infty$$

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Therefore, if we introduce space L_T (F_2) with f_1 (λ) $=\sqrt{f(\lambda) g(\lambda)}$, then the operators of the type in question, who appear with the representation

$$L_T(G) \xrightarrow{B_1} L_T(F_1) \text{ and } L_T(F_1) \xrightarrow{B_2} L_T(F),$$

under condition (6.15) are such, that $B^*_1 B_1$ and $B^*_2 B_2$ - the operators of Gil'berta - Schmidt. It is easy to see that operator $A^* A = B^*_1 B^*_2 B_2 B_1$ nuclear (for example, see [5], page 57).

§7. Some properties of trajectories [1]

FOOTNOTE [1]. It is in more detail with the questions, which are examined in this paragraph, it is possible to be introduced according to book [27]. ENDFOOTNOTE.

1. Differentiability on the average. Some asymptotic relationship/ratios. Let ξ (t), $-\infty < t < \infty$,-Gaussian stationary process with the continuous time t.

Process ξ (t) it is called that which is differentiated (on the

average), if there is a limit (on the average)

$$\lim_{h\to 0}\frac{\xi(t+h)-\xi(t)}{h}=\xi'(t).$$

The limit indicated exists when and only when in hilbert space L_T (F), where T = (-∞, ∞), there is the limit

$$\lim_{h\to 0}\frac{e^{i\lambda h}-1}{h}e^{i\lambda t}=i\lambda e^{i\lambda t},$$

that, obviously, equivalently to the condition

$$\int_{-\infty}^{\infty}\lambda^2F(d\lambda)<\infty \qquad (7.1)$$

(here F (dλ) - the spectral measure of stationary process ξ (t)).

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If

$$\xi(t)=\int_{-\infty}^{\infty}e^{i\lambda t}\Phi(d\lambda)$$

- the spectral representation of Gaussian stationary process ξ (t), then its derivative $\xi'(t),\ -\infty<t<\infty,$ also Gaussian stationary process - is

$$\xi'(t)=\int_{-\infty}^{\infty}e^{i\lambda t}(i\lambda)\Phi(d\lambda).$$

It is easy to see that condition (7.1) to the equivalently following:

$$M[\Delta_h\xi(t)]^2=\Delta_{-h}\Delta_hB(0)=O(h^2) \qquad (7.2)$$

with h $\rightarrow$ 0, where B (t) - correlation function, Λ and Δ_h indicates the operator of the taking of difference (for example, Δ_h B (t) = B (t + h) - B (t)). Actually, under condition (7.2) for any Λ

$$\int_{-\Lambda}^{\Lambda} \lambda^2 F(d\lambda) \leqslant C \int_{-\Lambda}^{\Lambda} \frac{1-\cos\lambda h}{h^2} F(d\lambda) \leqslant C \frac{\Delta_{-h}\Delta_h B(0)}{h^2}$$

for sufficiently small h (no matter how was greatly preassigned Λ), where C - certain constant, so that from condition (7.2) it escape/ensues (7.1).

Let us consider undifferentiable stationary process ξ (t). Let us explain, during which limitations on spectral measure F (dλ) occurs the relationship/ratio

$$\Delta_{-h}\Delta_h B(0) = O\{|h|^{2\alpha}\}, \qquad (7.3)$$

where $0 < \alpha < 1$.

We have

$$\frac{\Delta_{-h}\Delta_h B(0)}{h^{2\alpha}} = \int_{-\infty}^{\infty} \left(\frac{1-\cos\lambda h}{h^2}\right)^{\alpha} (1-\cos\lambda h)^{1-\alpha} F(d\lambda) \leqslant$$

$$\leqslant C \int_{-\infty}^{\infty} \lambda^{2\alpha} F(d\lambda),$$

so that condition (7.3) carried out, when

$$\int_{-\infty}^{\infty} \lambda^{2\alpha} F(d\lambda) < \infty. \qquad (7.4)$$

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Let us pause in more detail at the case, when there is spectral density f (λ) = F (dλ) /dλ. Let with sufficiently large $\lambda, |\lambda| \geqslant \Lambda,$

$$f(\lambda) = |\lambda|^{-\beta}$$

(where β > 1, since the spectral density f (λ) is the integrated($f\lambda$) function). For an undifferentiable stationary process β ⩽ 3. Let β < 3. Then

$$\frac{\Delta_{-h}\Delta_h B(0)}{h^{2\alpha}} = 2\int_{\Lambda}^{\infty} \frac{1-\cos\lambda h}{\lambda^{\beta} h^{2\alpha}}\, d\lambda + O\{h^{2(1-\alpha)}\}.$$

During the replacement of variable λh = μ we have :

$$\int_{\Lambda}^{\infty} \frac{1-\cos\lambda h}{\lambda^{\beta} h^{2\alpha}}\, d\lambda = h^{\beta-2\alpha-1} \int_{\Lambda h}^{\infty} \frac{1-\cos u}{u^{\beta}}\, du \sim$$
$$\sim h^{\beta-2\alpha-1} \int_{0}^{\infty} \frac{1-\cos h}{u^{\beta}}\, du$$

and, therefore,

$$\frac{\Delta_{-h}\Delta_h B(0)}{h^{2\alpha}} \sim C h^{\beta-2\alpha-1},$$

where

$$C = 2\int_{0}^{\infty} \frac{1-\cos u}{u^{\beta}}\, du.$$

FOOTNOTE [1]. Relationship/ratio $\alpha \sim \beta$ for variables α and β means that $\lim \alpha/\beta = 1$. ENDFOOTNOTE.

From the obtained relationship/ratios it is easy to deduce, that if

$$f(\lambda) = O\{|\lambda|^{-\beta}\}, \qquad (7.5)$$

that condition (7.3) is satisfied when $2\alpha = \beta - 1$; but if

$$\varlimsup_{\lambda\to\infty} f(\lambda)|\lambda|^{\beta} = \infty, \qquad (7.6)$$

that also when $2\alpha = \beta - 1$

$$\varlimsup_{h\to 0} \frac{\Delta_{-h}\Delta_h B(0)}{h^{2\alpha}} = \infty.$$

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Analogously for the spectral plane f (λ) of type (7.5) with $\beta = 3$ we obtain, that

$$\frac{\Delta_{-h}\Delta_h B(0)}{h^2} \leqslant C \int_{\Lambda h}^{\infty} \frac{1-\cos u}{u^3}\, du + O(1) = O\{|\ln|h||\},$$

and consequently,

$$\Delta_{-h}\Delta_h B(0) = O\{h^2|\ln|h||\}. \qquad (7.7)$$

2. Module/modulus of continuity. Let us consider undifferentiable Gaussian stationary process $\xi(t)$, $-\infty < t < \infty$, with the correlation function B (t), that satisfies condition (7.3).

Theorem 4. Under condition (7.3) there is an equivalent Gaussian process $\xi(t)$, for each trajectory of which with sufficiently small h is evenly on t in each finite interval

$$|\Delta_h\xi(t)| \leqslant C|h|^\alpha |\ln|h||^{1/2}, \qquad (7.8)$$

where C - certain constant.

Proof. With sufficiently small h we have

$$\begin{aligned} &\mathsf{P}\{|\Delta_h\xi(t)| > c'|h|^\alpha |\ln|h||^{1/2}\} \leqslant \\ &\leqslant \mathsf{P}\left\{\frac{|\Delta_h\xi(t)|}{|\Delta_{-h}\Delta_h B(0)|^{1/2}} > \frac{c'}{c''}|\ln|h||^{1/2}\right\} = \\ &= \frac{2}{\sqrt{2\pi}} \int\limits_{\frac{c'}{c''}|\ln|h||^{1/2}} e^{-x^2/2}\,dx \leqslant \frac{2}{\sqrt{2\pi}} \int\limits_{\frac{c'}{c''}|\ln|h||^{1/2}} x e^{-x^2/2}\,dx = \\ &= \frac{2}{\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{c'}{c''}\right)^2 \ln|h|} = \frac{2}{\sqrt{2\pi}}|h|^\beta, \quad \beta = \frac{1}{2}\left(\frac{c'}{c''}\right)^2, \end{aligned}$$

where c'' - constant in relationship/ratio $\Delta_{-h}\Delta_h B(0) \leqslant c''|h|^{2\alpha}$, and c' selected by such, that $\beta > 1$.

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Let us consider the initial process ξ (t) at the binary-rational points of form $t=k/2^n$, by counting for simplicity, that $0 \leq t \leq 1$. With $h = 2^{-n}$ we have

$$P\left\{\max_h \left|\Delta_h \xi\left(\frac{k}{2^n}\right)\right| > c' |h|^\alpha |\ln |h||^{1/2}\right\} \leqslant$$
$$\leqslant \sum_{k=0}^{2^n-1} \left(\frac{2}{\sqrt{2\pi}} |h|^\beta\right) = \frac{2}{\sqrt{2\pi}} 2^{-(\beta-1)n}.$$

Since $\beta > 1$ and a series $\sum_{n=1}^{\infty} 2^{-(\beta-1)n}$ descends, on Borel - Kantelli's known lemma

$$\left|\Delta_h \xi\left(\frac{k}{2^n}\right)\right| \leqslant c' |h|^\alpha |\ln ||h||^{1/2}$$

with sufficiently small h it is evenly on all $k = 0, \ldots, 2^n - 1$..

It is easy to see that any segment $[k/2^n, k_1/2^{n_1}]$ can be decomposed for the sum of segments $[r/2^m, (r+1)/2^m]$, where r, m are integers, whereupon with any m it is possible to take in this sum not more than two segments of the form indicated. Thus, for any h (with proper n, included within limits $2^{-n} \leq h \leq 2^{-n+1}$)

$$h = \frac{k_1}{2^{n_1}} - \frac{k}{2^n} = \sum_m{}' 2^{-m},$$

where $\sum^{*}$ designates summing over corresponding m. Accepting this into consideration, for sufficiently small h we obtain

$$\left|\Delta_h\xi\left(\frac{k}{2^n}\right)\right| \leqslant \sum_m^{*}\left|\Delta_{2^{-m}}\xi\left(\frac{r}{2^m}\right)\right| \leqslant$$
$$\leqslant \sum_m^{*} c' 2^{-\alpha m}|\ln 2^m|^{1/2} \leqslant$$
$$\leqslant 2^{-\alpha n}(\ln 2^n)^{1/2}\, 2c' \sum_{m=n}^{\infty} 2^{-\alpha(m-n)}\left(\frac{\ln 2^m}{\ln 2^n}\right)^{1/2} \leqslant$$
$$\leqslant |h|^{\alpha}|\ln|h||^{1/2}\, 2c' \sum_{k=0}^{\infty} 2^{-\alpha k}\left(1+\frac{k}{n}\right)^{1/2} \leqslant$$
$$\leqslant C|h|^{\alpha}|\ln|h||^{1/2}.$$

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Thus, with probability 1 trajectories of the process $\xi(t)$ in question) they satisfy condition (7.8) on many all binary-rational points; specifically, with almost all $\omega \in \Omega$ trajectory $\xi(\omega, \cdot) = \xi(\omega, t)$ are evenly continuous functions on the everywhere dense set of the binary-rational points t_{kn}. It is obvious, for the arbitrary point t there is limit $\lim_{t_{kn}\to t} \xi(\omega, t_{kn})$, which for almost all $\omega \in \Omega$

coincides with original value ξ (t) = ξ (ω, t). It is clear that for an equivalent process with the values, defined as

$$\xi(\omega, t) = \lim_{t_{kn} \to t} \xi(\omega, t_{kn}),$$

with almost all ω the trajectories satisfy condition (7.8). Redefining for the others ω of value ξ (ω, t) (after assuming, for example, ξ (ω, t) $\equiv$ 0), we will obtain the equivalent process ξ (t), all trajectories of which already they satisfy condition (7.8). Theorem is demonstrated.

3. Some limit theorems. Let us consider undifferentiable Gaussian stationary process ξ (t) with correlation function B (t). Let us assume that in interval (0, τ) there is (with the exception of the finite number of points) a second derivative $B''(t)$, that has only first-order discontinuities, i.e., for any point t within interval (0, τ) there are final limits $B''(t-0) = \lim_{h \to 0}$ B (t-h) and B" (t + 0) = $\lim_{h \to 0}$ B (t + h). Recall (see Section 1) that for an undifferentiable process ξ (t)

$$\lim_{h \to 0} \frac{\Delta_{-h} \Delta_h B(0)}{h^2} = \infty, \qquad (7.9)$$

so that at point t = 0 derived B" (t) has second-order discontinuity/interruption, more precise,

$$\lim_{h \to 0} B''(h) = -\infty.$$

so that in points t = 0 derived B" (t) has second-order discontinuity/interruption, more precise,

Theorem 5. Occurs the following asymptotic relation:

$$\lim_{h\to 0}\frac{1}{N}\sum_{k=0}^{N-1}\frac{[\Delta_h\xi(kh)]^2}{\Delta_{-h}\Delta_h B(0)}=1, \qquad (7.10)$$

where h = τ/N, and limit is understood in the sense of mean convergence.

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Proof. Let n be a number of points of discontinuity of a function B" (t). Let us take as small as desired $\varepsilon > 0$. Each point of discontinuity t, $0 \leqslant t < \tau$, can be surrounded by certain interval so that their general length would not exceed ε. Let us designate I_ε addition to the association of these intervals (I_ε - the association of the finite number of segments). It is obvious, function B" (t) is evenly continuous on set I_ε and

$$\Delta_{-h}\Delta_h B(t) = O\{B''(t)h^2\} \qquad (7.11)$$

evenly on $t \in I_\varepsilon$. Taking into account relationship/ratio (7.7), we obtain, which

$$\Delta_{-h}\Delta_h B(t) = o\{\Delta_{-h}\Delta_h B(0)\} \qquad (7.12)$$

is evenly on $t \in I_{\varepsilon}$. Furthermore, since together with the correlation function B (t) positively determined is the function $\Delta_{-h}\Delta_h B(t)$, with all t

$$|\Delta_{-h}\Delta_h B(t)| \leqslant |\Delta_{-h}\Delta_h B(0)|. \tag{7.13}$$

We have

$$M\Delta_h\xi(s)\,\Delta_h\xi(t) = \Delta_{-h}\Delta_h B(s-t)$$

also, according to common/general/total formula (5.10)

$$\begin{aligned} M[\Delta_h\xi(s)\cdot\Delta_h\xi(t)\cdot\Delta_h\xi(u)\cdot\Delta_h\xi(v)] &= \\ = \Delta_{-h}\Delta_h B(s-t)&\cdot\Delta_{-h}\Delta_h B(u-v) + \\ + \Delta_{-h}\Delta_h B(s-u)&\cdot\Delta_{-h}\Delta_h B(t-v) + \\ + \Delta_{-h}\Delta_h B(s-v)&\cdot\Delta_{-h}\Delta_h B(t-u). \end{aligned}$$

It is easy to count, that

$$\begin{aligned} \sigma^2(h) &= M\left[\frac{1}{N}\sum_{k=0}^{N-1}\frac{\Delta_h\xi(kh)^2}{\Delta_{-h}\Delta_h B(0)} - 1\right]^2 = \\ &= \frac{2}{N^2}\sum_{k,j=0}^{N-1}\left[\frac{\Delta_{-h}\Delta_h B((k-j)h)}{\Delta_{-h}\Delta_h B(0)}\right]^2. \end{aligned} \tag{7.14}$$

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With fixed/recorded j to each interval of length δ can belong no more than $1 + [\delta h^{-1}]$ points of the form $(k - j)h$, and in all such points - is not more $N(1 + [\delta h^{-1}])$. Consequently, the number of points of

form (k - j) h, entering in addition to above set I_ε indicated, does not exceed number $(Nn + N^2\varepsilon/\tau)$. From relationship/ratios (7.11) - (7.14) we obtain, that, whatever $\varepsilon > 0$, for sufficiently small h

$$\sigma^2(h) \leqslant \frac{2}{N^2}\left(Nn + N^2\frac{\varepsilon}{\tau}\right) + 2\left\{\max_{t \in I_\varepsilon}\left[\frac{\Delta_{-h}\Delta_h B(t)}{\Delta_{-h}\Delta_h B(0)}\right]^2\right\} \leqslant C\varepsilon,$$

where C - certain constant. Theorem 5 is demonstrated.

Let us note that there is subsequence $h_1, h_2, \ldots$, for which asymptotic relation (7.10) is fulfilled with probability 1. Moreover, as $h_1, h_2, \ldots$ it is possible to take any sequence, for which

$$\sum_{n=1}^{\infty} \sigma^2(h_n) < \infty, \qquad (7.15)$$

since under condition (7.15) there is a sequence $\varepsilon_n \to 0$ such, which

$$\sum_{h=h_n} P\left\{\left|\frac{1}{N}\sum_{k=0}^{N-1}\frac{[\Delta_h\xi(kh)]^2}{\Delta_{-h}\Delta_h B(0)} - 1\right| \geqslant \varepsilon_n\right\} < \infty,$$

and on Borel - Kantelli's lemma hence follows that in (7.10) when $h = h_n$ occurs the convergence with probability 1.

In connection with this let us consider the question concerning the speed of decrease determined in (7.14) function σ^2 (h) with $h \to 0$:

$$\sigma^2(h) = \frac{2}{N^2}\frac{h^2}{[\Delta_{-h}\Delta_h B(0)]^2}\sum_{k,j=0}^{N-1}[\Delta_{-h}\Delta_h B((k-j)h)]^2.$$

Lemma 4. Under the done previously assumptions relative to function B" (t) occurs the following estimate:

$$d^2(h) = \sum_{k,\, j=0}^{N-1} [\Delta_{-h}\Delta_h B((k-j)h)]^2 =$$
$$= O\{\max(|\Delta_{-h}\Delta_h B(0)|,\ |\Delta_{-h}\Delta_h B(0)|^2 |h|^{-1})\}. \quad (7.16)$$

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Proof. It is obvious,

$$d^2(h) = O\left\{h^{-1}[\Delta_{-h}\Delta_h B(0)]^2 + h^2 \iint\limits_{|s-t|>2h} [B''(s-t)]^2\, ds\, dt\right\}.$$

Further, with any fixed $\delta > 0$

$$\iint\limits_{|s-t|>2h} [B''(s-t)]^2\, ds\, dt = O\left\{\int_{2h}^{\delta} B''(t)^2\, dt\right\},$$

and if in certain vicinity $(0, \delta)$ function $B''(t)$ is monotonic, then

$$\int_{2h}^{\delta} B''(t)^2\, dt = B''(2h+\theta)[B'(\delta) - B'(2h)],$$

where $2h \leqslant \theta \leqslant \delta$ and

$$B''(2h+\theta) = O\{h^2\Delta_{-h}\Delta_h B(0)\}.$$

If function $B''(t)$ is monotonic, then in certain vicinity $(0, \delta)$ it retains sign, so that $B'(t)$ is also monotonic. It is obvious, monotonic will be the function

$$\Delta_h B(t) = \int_t^{t+h} B'(s)\, ds.$$

Therefore, when $|B'(2h)| \to \infty$ with $h \to 0$, then

$$|B'(2h)| = O\{h^{-1}[\Delta_h B(h)]\}.$$

where because of the monotonicity of function $\Delta_h B(t)$

$$|\Delta_h B(h)| \leqslant |\Delta_h B(0)| = \frac{1}{2} |\Delta_{-h} \Delta_h B(0)|.$$

Thus,

$$\iint\limits_{|s-t|>2h} [B''(s-t)]^2\, ds\, dt = \\ = O \begin{cases} h^{-2} |\Delta_{-h} \Delta_h B(0)|, & \text{если} \quad B'(2h) \text{ ограничена; } (1) \\ h^{-3} |\Delta_{-h} \Delta_h B(0)|^2, & \text{если} \quad B'(2h) \text{ не ограничена. } (2) \end{cases}$$

Key: (1). if B' (2h) is limited. (2). if B' (2h) is not limited.

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In summation, we obtain, that

$$d^2(h) = O\{\max(|h|^{-1} |\Delta_{-h} \Delta_h B(0)|^2, |\Delta_{-h} \Delta_h B(0)|)\}.$$

Obtained relationship/ratio (7.16) gives the following estimation for function σ^2 (h):

$$\sigma^2(h) = O\left\{\max\left(|h|, \frac{h^2}{|\Delta_{-h} \Delta_h B(0)|}\right)\right\}. \qquad (7.17)$$

Specifically, with condition (7.3) the function σ^2 (h) decreases with h $\rightarrow$ 0 as certain degree $|h|^{\beta}$:

so that condition (7.15) will be implemented, for example, for any sequence of form $h_n = 2^{-n}$, n = 1, 2, ...

For the type of stationary Gaussian processes in question occurs also the following result, which supplements theorem 5.

Theorem 6. When

$$\Delta_{-h}\Delta_h B = o\{|h|^{1/2}\}, \tag{7.18}$$

the correctly asymptotic relation

$$\lim_{h\to 0} h^{-1}\frac{1}{N}\sum_{k=0}^{N-1}\Delta_h\xi(kh)\,\Delta_h\xi(t+kh) = B'(t-0) - B'(t+0), \tag{7.19}$$

where t is any fixed/recorded point of interval (0, τ), of N = [h^{-1} (τ - t)] - 1 and means convergence on the average.

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Proof. Such elementary calculations, as carried out are earlier, they show that value

$$\eta(h) = h^{-1}\frac{1}{N}\sum_{k=0}^{N-1}\Delta_h\xi(kh)\,\Delta_h\xi(t+kh)$$

has the average value

$$M\eta(h)=\frac{\Delta_h\Delta_{-h}B(t)}{h}=\frac{B(t-h)-B(t)}{h}-\frac{B(t+h)-B(t)}{h},$$

whereupon its dispersion is such, which

$$D\eta(h)\leqslant Cd^2(h).$$

From estimations (7.16) for value d^2 (h) is evident that under condition (7.18) occurs relationship/ratio (7.19).

Certainly for fast enough descending sequence $h = h_n$, n = 1, 2, ..., this relationship/ratio will be correct, also, with probability 1.

end section.

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Chapter II

Structure of spaces H (T) AND $L_T(F)$.

§1. Some preliminary information and results.

1. Introduction. Above (in §6 chapter I) we saw that the Hilbert space of random variables H (T), generated by the values of stationary process ξ (t), $t \in T$ (with the spectral measure F (dλ)), it is isometric to function space $L_T(F)$ - to the locked linear closure of function $e^{i\lambda t}$ of $\lambda \in [-\pi, \pi]$ in the case of discrete t, $\lambda \in (-\infty, \infty)$- in the case continuous t. Thereby appears possibility to trace stationary processes with the help of analytical means. For this useful to first study in more detail the analytical structure of spaces $L_T(F)$, than we will be occupied in this chapter, after being

bounded to those by the case, when T - interval or is half-line.

It is clear that it is possible to be bounded to interval of T = [-τ, τ] either T = [0, τ] and to semidirect T = (-∞, 0], T = [0, ∞), since the arbitrary interval or the half-line T_1 is obtained by the "shift/shear" of T to certain real t, and the corresponding space $L_{T_1}(F)$ is obtained from $L_T(F)$ by multiplication on $e^{i\lambda t}$.

In order that reader could more easily visualize, which results here one should expect let us assume that ξ (t) there is a stationary process with discrete time and spectral density f (λ) = 1. It is understandable that of this case $L_T(F)$ it consists of trigonometric polynomials $P(e^{i\lambda}) = \sum_{t \in T} c(t) e^{i\lambda t}$, if T - the finite interval, and actually coincides with noted Hardy space Hardy $\mathscr{H}^2$ in circle (or outside circle), if T - is half-line, more precise, $L_T(F)$ consists of the integrated squared functions φ (λ), which are decompose/expanded into one-sided Fourier series:

$$\varphi(\lambda) = \sum_{-\infty}^{0} c(t) e^{i\lambda t} \quad \text{при}^{(1)} \quad T = (-\infty, 0]$$
$$\left(\text{или}^{(2)} \ \varphi(\lambda) = \sum_{0}^{\infty} c(t) e^{i\lambda t} \ \text{при}^{(1)} \ T = [0, \infty)\right).$$

Key: (1). with. (2). or.

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What by itself does represent space $L_T(F)$, if f (λ) ≠ 1, if are examined processes with continuous time?

From the very beginning usefully to note that if the spectral measures F (dλ) and G (dλ) were connected by the inequality

$$F(d\lambda) \geqslant G(d\lambda)$$

(F (dλ) it majorizes G (dλ)), then appropriate hilbert spaces $L_T(F)$ and $L_T(G)$ they satisfy the connection/inclusion

$$L_T(F) \subseteq L_T(G).$$

This obvious fact is the corollary of the fact that any fundamental sequence of the functions of form $\varphi_n(\lambda) = \sum_k c_{kn} e^{i\lambda t_{kn}}$, n = 1, 2, ..., in space $L_T(F)$, that converges to function $\varphi(\lambda) \in L_T(F)$, simultaneously is fundamental, also, in space $L_T(G)$:

$$\|\varphi_m - \varphi_n\|_G \leqslant \|\varphi_m - \varphi_n\|_F \to 0$$

with m, n → ∞, the limit function $\psi(\lambda) \in L_T(G)$ coinciding almost everywhere relative to G (dλ) with the above indicated by limit function $\varphi(\lambda) \in L_T(F)$. Consequently, functions ϕ (λ) and $\psi(\lambda)$ coincide as cell/elements the hilbert space $L_T(G)$, i.e., $\varphi(\lambda) \in L_T(G)$.

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Hence immediately it follows that under the condition, when

there is spectral density $f(\lambda) = F(d\lambda)/d\lambda$ of the type [1]

$$f(\lambda) \asymp 1 \qquad (1.1)$$

(in the case of the discrete time t), just as with $f(\lambda) \equiv 1$, space $L_{[0,\tau]}(F)$ consists of all polynomials

$$P(e^{i\lambda}) = \sum_{t=0}^{\tau} c(t)\, e^{i\lambda t}$$

(with the real coefficients of $c(t)$, $0 \leqslant t \leqslant \tau$), and space $L_{(-\infty,0]}(F)$ ($L_{[0,\infty)}(F)$)consist of the integrated squared functions, representable by Fourier series of the form

$$\varphi(e^{i\lambda}) = \sum_{-\infty}^{0} c(t)\, e^{i\lambda t} \qquad \left(\varphi(e^{i\lambda}) = \sum_{0}^{\infty} c(t)\, e^{i\lambda t}\right),$$

and coinciding with the boundary values (with $r \to 1$) analytic functions $\varphi(z)$ of circle $|z|<1$ (outside circle) Hardy's mentioned above class $\mathcal{H}^2$:

$$\varphi(e^{i\lambda}) = \lim_{r\to 1} \varphi(z), \qquad z = re^{i\lambda}.$$

In case of continuous time t analogous to condition (1.1) is condition

$$f(\lambda) \asymp (1+\lambda^2)^{-n}. \qquad (1.2)$$

FOOTNOTE [1]. Recall that for variables α and β relationship/ratio $\alpha \asymp \beta$ it designates, that $0 \leqslant c_1 \leqslant \alpha/\beta \leqslant c_2 < \infty$. ENDFOOTNOTE.

with n - natural number.

With n = 0, determined of §6 Chapter I space $L_{[0,\tau]}(F)$, obviously,

is of the integrated squared functions ϕ (λ), representable the integral of Fourier of form $\varphi(\lambda)=\int_0^\tau e^{i\lambda t}c(t)\,dt$, and it is analogous space $L_{(-\infty,0)}(F)$ (and $L_{[0,\infty)}(F)$) they consist of the functions of form ϕ (λ) = $=\int_{-\infty}^{0} e^{i\lambda t}c(t)\,dt$ $\left(\varphi(\lambda)=\int_0^\infty e^{i\lambda t}c(t)\,d(t)\right)$.

Using only basic information, let us show that under condition (1.2) the space $L_{[0,\tau]}(F)$ coincides with the class of functions, representable in the form

$$\varphi(\lambda)=P(i\lambda)+(1+i\lambda)^n\int_0^\tau e^{i\lambda t}c(t)\,dt, \qquad (1.3)$$

where P (iλ) - the polynomial of degree not higher than n - 1 (with real coefficients)), but c (t) - the integrated squared (real) function.

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In fact, to space $L_{[0,\tau]}(F)$ belong all the functions $\varphi(\lambda)=(i\lambda)^k e^{i\lambda s}$, k = 1, ..., n - 1, that are the limit of the form

$$\varphi(\lambda)=\lim_{h\to 0}(i\lambda)^{k-1}\frac{e^{i\lambda(s+h)}-e^{i\lambda s}}{h},$$

so that any polynomial $P(i\lambda)=\sum_0^{n-1}c_k(i\lambda)^k$ enters in $L_{[0,\tau]}(F)$. Further, entering also in $L_{[0,\tau]}(F)$ function ϕ (λ) $=(1+i\lambda)^{n-1}(e^{(1+i\lambda)s}-1)$, $0\leqslant s\leqslant\tau$, represented in the form

$$\varphi(\lambda) = (1 + i\lambda)^n \int_0^\tau e^{i\lambda t} c_s(t)\, dt,$$

where

$$c_s(t) = \begin{cases} e^t \text{ (1) при } & 0 \leqslant t \leqslant s, \\ 0 \text{ (1) при } & s \leqslant t \leqslant \tau. \end{cases}$$

Key: (1). with.

It is easy to see that the locked linear closure of functions $\varphi(\lambda) = (1 + i\lambda)^{n-1}$ $(e^{(1+i\lambda)s} - 1)$, $0 \leqslant s \leqslant \tau$, and $\varphi(\lambda) = (i\lambda)^k$, 0 ≤ k ≤ n - 1, gives whole space $L_{[0,\tau]}(F)$ (regarding being locked linear shell functions $e^{i\lambda s}$, $0 \leqslant s \leqslant \tau$), since, being transmitted of the indicated functions $\varphi(\lambda)$, by consecutive integration can lead to functions $\varphi(\lambda) = e^{i\lambda s}$; let us say,

$$\int_0^t (1 + i\lambda)^{n-1} e^{(1+i\lambda)s}\, ds = (1 + i\lambda)^{n-2} (e^{(1+i\lambda)t} - 1) \text{ (1) и т. д.}$$

Key: (1). and so forth.

It is obvious also that the linear closure of "stepped" function $c_s(t)$ the form indicated, where parameter s passes entire segment [0, τ], is everywhere dense in hilbert space $\mathscr{L}^2[0, \tau]$ the integrated squared functions c (t), 0 ≤ t ≤ τ.

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Furthermore, for functions φ' (λ) and φ" (λ) the form

$$\varphi(\lambda)=(1+i\lambda)^n\int\limits_0^{\tau} e^{i\lambda t}c(t)\,dt,$$

where c (t) is the linear combination of step functions $c_s(t)$, on the known equation of Perseval

$$\|\varphi'(\lambda)-\varphi''(\lambda)\|_F^2=\int\limits_{-\infty}^{\infty}|\varphi'(\lambda)-\varphi''(\lambda)|^2 f(\lambda)\,d\lambda \asymp$$
$$\asymp\int\limits_{-\infty}^{\infty}|\varphi'(\lambda)-\varphi''(\lambda)|^2|1+i\lambda|^{2n}\,d\lambda=2\pi\int\limits_0^{\tau}|c'(t)-c''(t)|^2\,dt.$$

It is clear that the locked linear closure of all functions ϕ (λ) the form indicated coincides with the class of functions ϕ (λ), representable by the formula

$$\varphi(\lambda)=(1+i\lambda)^n\int\limits_0^{\tau} e^{i\lambda t}c(t)\,dt,$$

where $c(t)\in\mathscr{L}^2[0,\tau]$. After connecting up it all the functions $\varphi(\lambda)=(i\lambda)^k$, $0\leqslant k\leqslant n-1$, we we will obtain, obviously, space $L_{[0,\tau]}(F)$.

Formula (1.3) makes it possible to give the very demonstrative description of the values from space H (T) at T = [0, τ] - the locked linear closure of values ξ (t), 0 ≤ t ≤ τ. Specifically, if $\Phi(d\lambda)$ is a stochastic spectral measure of stationary process ξ (t), then is any value $\eta\in \dot{H}(T)$ represented by integral $\eta=\int\varphi(\lambda)\Phi(d\lambda)$ (see §6 of

Chapter I), where $\varphi(\lambda) \in L_T(F)$, and is easy to see that

$$\eta = \sum_{k=0}^{n-1} \left[a_k \xi^{(k)}(0) + b_k \xi^{(k)}(\tau) + \int_0^\tau \xi^{(k)}(t) c_k(t)\, dt \right],$$

where a_k and b_k are some real coefficients, $c_k(t)$ are the integrated squared functions, and $\xi^{(k)}(t)$ - the available at process derivatives, k = 0, ..., n - 1.

Let us note that if spectral density f (λ) satisfies only the condition

$$f(\lambda) \geqslant c(1+\lambda^2)^{-n},$$

that appropriate space $L_{[0,\tau]}(F)$ it enters in the space $L_T(G)$, which corresponds to spectral density $g(\lambda) = \phi(1+\lambda^2)^{-n}$, so that any function $\varphi(\lambda) \in L_{[0,\tau]}(F)$ represented by formula (1.3).

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It is important to note also that this formula with all composite λ assigns whole analytic function. In future, (see §4 of Chapter III) we will show that space $L_{[0,\tau]}(F)$ can be identified with the class of the functions of form (1.3) not only under condition (1.2), but also under the weaker condition, when

$$f(\lambda) \asymp (1+\lambda^2)^{-n}, \quad \lambda \to \infty$$

(i.e. $f(\lambda)\asymp(1+\lambda^2)^{-n}$ only with sufficiently large λ). Under condition (1.2) from representation (1.3) it is easy to deduce common/general/total formula for functions φ (λ) from spaces $L_{(-\infty,0]}(F)$ and $L_{[0,\infty)}(F)$. Specifically, any function $\varphi(\lambda)\in L_{[0,\infty)}(F)$ is a limit of certain sequence of functions $\varphi_k(\lambda)\in L_{[0,\tau_k]}(F), \tau_k\to\infty$, representable in the form

$$\varphi_k(\lambda)=P_k(i\lambda)+(1+i\lambda)^n\int_0^\infty e^{i\lambda t}c_k(t)\,dt,\quad k=1,2,\ldots,$$

where the sequence $c_k(t)$, $k=1, 2, \ldots$, is fundamental in hilbert space $\mathscr{L}^2(0,\infty)$ the integrated squared functions it descends in this space to certain function c (t), $0 \leqslant t \leqslant \infty$. It is clear that the limit function $\varphi(\lambda)=\lim\limits_{k\to\infty}\varphi_k(\lambda)$ represented in the form

$$\varphi(\lambda)=P(i\lambda)+(1+i\lambda)^n\int_0^\infty e^{i\lambda t}c(t)\,d(t),$$

where $P(i\lambda)=\lim P_k(i\lambda)$ - the polynomial of degree is not higher than n - 1. It is clear also, that everyone such functions (where $c(t)\in\mathscr{L}^2(0,\infty)$) enters in space $L_{[0,\infty)}(F)$.

Analogously space $L_{(-\infty,0]}(F)$ coincides with the class of all functions, described by the formula

$$\varphi(\lambda)=P(i\lambda)+(1+i\lambda)^n\int_{-\infty}^0 e^{i\lambda t}c(t)\,d(t).$$

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Will be traced below the structure of corresponding spaces $L_T(F)$, when spectral density f (λ) compulsorily does not satisfy condition (1.2), but all the same it decreases at infinity (or it is converted in 0 with final λ) "not too strongly", namely when

$$\int_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda > -\infty$$

for the discrete time t and

$$\int_{-\infty}^{\infty} \frac{\ln f(\lambda)}{1+\lambda^2}\, d\lambda > -\infty$$

for the continuous time t. In this case, after retaining the previous designations, we will turn to the composite spaces H (T) and $L_T(F)$.

2. Functions, analytical in circle. Let us designate by $\mathcal{H}^p$, $1 \leqslant p \leqslant \infty$, the class analytic functions ϕ (z) in unit circle $|z|<1$, for which

$$\lim_{r\to 1} \int_{-\pi}^{\pi} |\varphi(re^{i\lambda})|^p\, d\lambda < \infty,\ z = re^{i\lambda}.$$

If $\varphi \in \mathcal{H}^p$, then for almost all $\lambda \in [-\pi, \pi]$ there are boundary values $\varphi(e^{i\lambda}) = \lim_{r\to 1} \varphi(re^{i\lambda})$ and

$$\lim_{r\to 1} \int_{-\pi}^{\pi} |\varphi(re^{i\lambda})|^p\, d\lambda = \int_{-\pi}^{\pi} |\varphi(e^{i\lambda})|^p\, d\lambda.$$

Space $\mathcal{H}^p$ Banach space with norm $\|\varphi\|^{(p)} = \left(\int_{-\pi}^{\pi} |\varphi(e^{i\lambda})|^p\, d\lambda\right)^{1/p}$. It is possible to identify $\mathcal{H}^p$ with the locked subspace (in known space $\mathcal{L}^p(-\pi, \pi)$) all functions $\varphi(e^{i\lambda}) \in \mathcal{L}^p(-\pi, \pi)$, for which

$$\int_{-\pi}^{\pi} \varphi(e^{i\lambda}) e^{in\lambda} d\lambda = 0, \qquad n = 1, 2, \ldots$$

FOOTNOTE [1]. Space $\mathscr{L}^p(a, b)$ consists of functions ϕ (λ) on segment $a \leqslant \lambda \leqslant b$, for which $\|\varphi\|^{(p)} = \left(\int_a^b |\varphi(\lambda)|^p d\lambda\right)^{1/p} < \infty$. ENDFOOTNOTE.

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This subspace, which consists of the boundary values of the described above functions, analytical in circle, also let us designate $\mathscr{H}^p$.

Function $\varphi(z)$ analytical within range $|z| < 1$ is called an external function if it is represented in the form

$$\varphi(z) = a \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\lambda} + z}{e^{i\lambda} - z} \ln p(\lambda) d\lambda\right\}, \quad |a| = 1,$$

where real function $p(\lambda)$ is non-negative and $\ln p \in \mathscr{L}^1(-\pi, \pi)$.

Analytic within circle function ϕ (z) is called internal, if $|\varphi(z)| \leqslant 1$ and $|\varphi(e^{i\lambda})| = 1$ for almost all $\lambda \in [-\pi, \pi]$.

Blaschke's product is called analytic function B (z) of the form

$$B(z) = az^p \prod \left[\frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z}\right]^{p_n}, \quad |a| = 1,$$

where p, p_1, p_2, ... - nonnegative integers, $0 \leqslant |a_n| < 1$ and product $\prod |a_n|^{p_n}$ descends.

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Theorem ([10], pp. 98-99). Internal function ϕ (z) in an only manner is represented in the form of the product

$$\varphi(z) = B(z)\exp\left\{-\int_{-\pi}^{\pi}\frac{e^{i\lambda}+z}{e^{i\lambda}-z}\mu(d\lambda)\right\},$$

where B (z) - the function of Blaschke, and μ (dλ) - singular measure.

From this result easily is derive/concluded) ([10], page 123), that any non-empty family of internal functions has the greatest common/general/total (internal) divider/denominator.

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Let us designate by D that introduced by V. I. Smirnov (see [20]) the class analytic in circle $|z|<1$ functions ϕ (z), which allow/assume the representation

$$\varphi(z) = B(z)\exp\left\{-\int_{-\pi}^{\pi}\frac{e^{i\lambda}+z}{e^{i\lambda}-z}\mu(d\lambda)\right\}\times$$
$$\times\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\lambda}+z}{e^{i\lambda}-z}\ln p(\lambda)\,d\lambda\right\}, \quad (1.4)$$

where B (z) - Blaschke's product, μ (dλ) - singular measure and p (λ) $\geqslant 0$, $\ln p \in \mathscr{L}^1(-\pi,\pi)$. Thus, class D consists of the functions ϕ (z), representable in the form of the product of certain internal function

(internal part ϕ) and of certain external function (exterior ϕ).

For each function ϕ (z) class D are with almost all λ boundary values $\varphi(e^{i\lambda})=\lim_{r\to 1}\varphi(re^{i\lambda})$, which satisfy condition $|\varphi(e^{i\lambda})|=p(\lambda)$, where p (λ) - the function, which figures in representation (1.4) for ϕ (z).

Theorem ([10], p 80). All functions $\varphi\in\mathscr{H}^1$ there is a product of two functions from $\mathscr{H}^2$.

Theorem ([10], p 81). With $f(\lambda)>0$ и $f\in\mathscr{L}^1$ function $f=|\varphi|^2$, where $\varphi\in\mathscr{H}^2$ in that and only that case, if $\ln f\in\mathscr{L}^1$.

Apropos of this theorem let us note that it is possible to take

$$\varphi(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\lambda}+z}{e^{i\lambda}-z}\ln f(\lambda)\,d\lambda\right\}.$$

The theorem of Byerling ([10], p 145). During $\varphi\in\mathscr{H}^2$ function $\{z^n\varphi\}$, n = 0, 1, ..., they generate entire class $\mathscr{H}^2$ in that and only that case, if ϕ is an external function.

Let ϕ (z) - function, is analytical in circle. In that case the function ϕ (1/z) is analytical outside circle. After comparing thus of each function, analytical in circle, the function, analytical

outside circle, we we will obtain the classes of functions D and $\mathcal{H}^p$, analytical outside circle.

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In order to differ these classes from each other, we will in the case of necessity designate the appropriate classes within circle D^+, $\mathcal{H}^{p+}$, outside circle - D^-, $\mathcal{H}^{p-}$.

3. Functions, analytical in half-plane. Let us designate by $\tilde{\mathcal{H}}^p$, D the classes of functions, analytical in the upper half-plane, which are the forms of classes $\mathcal{H}^p$, D in circle with the conformal mapping of circle onto the upper half-plane.

By $\mathcal{H}^p$ let us designate the class of those analytic in the upper half-plane functions ϕ (z), for which

$$\int_{-\infty}^{\infty} |\varphi(x+iy)|^p\, dx \leqslant M < \infty, \quad y \geqslant 0,$$

where constant M does not depend on y ([10], [15]).

Function $\varphi \in D$ is called external, if it is is represented in the form

$$\varphi(z) = \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+\lambda z}{\lambda - z}\,\frac{\ln p(\lambda)}{1+\lambda^2}\, d\lambda\right\},$$

where p (λ) - real function, p (λ) $\geqslant$ 0, and $\frac{\ln p(\lambda)}{1+\lambda^2} \in \mathcal{L}^1(-\infty, \infty)$. For external functions in inequality for an integral of Poisson

$$\ln|\varphi(z)| \leqslant \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\ln|\varphi(\lambda)|}{(\lambda-x)^2+y^2} d\lambda, \ z = x+iy, \ y>0,$$

occurs the equality.

Function $\varphi \in D$ is called internal, if $|\varphi(z)| \leqslant 1, \ |\varphi(\lambda)| = 1 \ (z = \lambda + i\mu, \ \mu \geqslant 0)$.

For the functions of class D they occur of representation of the form (1.4):

$$\varphi(z) = e^{i\alpha} B(z) \exp\left\{-\int_{-\infty}^{\infty} \frac{1+\lambda z}{\lambda - z} \mu(d\lambda)\right\} \times$$
$$\times \exp\left\{\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1-\lambda z}{\lambda - z} \frac{\ln p(\lambda)}{1+\lambda^2} d\lambda\right\},$$

where α are real number, B (z) is a function of Blaschke, μ (dλ) - singular final measure and p (λ) $\geqslant$ 0, $\frac{\ln p(\lambda)}{1+\lambda^2} \in \mathcal{L}^1(-\infty, \infty)$.

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Analogously to classes D and $\mathcal{H}^p$ in the upper half-plane extend remaining assertions. Specifically, occurs the following.

Theorem of Lax ([10]). During $\varphi \in \mathscr{H}^2$ function $\{e^{i\lambda t}\varphi(\lambda), t \geqslant 0\}$ they generate everything $\mathscr{H}^2$ in that and only that case if ϕ is an external function.

Below we frequently will use the following characteristic of spaces $\mathscr{H}^2$.

Theorem of Pali - Weiner ([10]), p 187). Function $\varphi \in \mathscr{H}^2$ (in the upper half-plane) in that and only that case, if

$$\varphi(z) = \int_0^\infty e^{izt} c(t)\, dt, \quad \operatorname{Im} z \geqslant 0,$$

where $c(t) \in \mathscr{L}^2(0, \infty)$.

If simultaneously with classes D, $\mathscr{H}^p$ in the upper half-plane are considered classes D, $\mathscr{H}^p$ in lower half-plane, we will write D^+, $\mathscr{H}^{p+}$ in the case of the upper half-plane and D^-, $\mathscr{H}^{p-}$ in the case lower. Let us note that

$$\mathscr{L}^2(-\infty, \infty) = \mathscr{H}^{2+} \oplus \mathscr{H}^{2-}.$$

§2. Spaces $L^+(F)$ and $L^-(F)$

Let ξ (t) be a stationary (in the broad sense) random process with the spectral measure F (dλ). Let $F=F_a+F_s$, where F_a is [redacted] absolutely continuous and F_s is singular components of measure F. Let us assume, $f(\lambda)=\frac{dF_a}{d\lambda}$ let us in this case call f (λ) spectral density, even if $F\neq F_a$. Let us introduce for a multiplicity new designations, after assuming

$$L=L(F)=L_{(-\infty,\,\infty)}(F),$$
$$L^-=L^-(F)=L_{(-\infty,\,0]}(F)\ \text{and}\ L^+=L^+(F)=L_{[0,\,\infty)}(F).$$

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Theorem 1. If ξ (t) is a stationary random process with the discrete time t = 0, ±1, ..., then

1) $L^+(F_s)=L^-(F_s)=L(F_s)$;

2) $L^+(F_a)=L^-(F_a)=L(F_a)$ in that and only that case, if

$$\int_{-\pi}^{\pi}|\ln f(\lambda)|\,d\lambda=\infty;$$

3) if

$$\int_{-\pi}^{\pi}|\ln f(\lambda)|\,d\lambda<\infty, \qquad (2.1)$$

that f (λ) can be registered in the form $f(\lambda)=|g(e^{i\lambda})|^2$, where g is an external function of class $\mathcal{H}^2$ in circle $|z|<1$. In this case

$$L^{-}(F_a) = D^{-} \cap L(F_a) = \frac{1}{g}\mathcal{H}^{2-}, \quad L^{+}(F_a) = D^{+} \cap L(F_a) = \frac{1}{g}\mathcal{H}^{2+}.$$

Theorem 2. If ξ (t) is a stationary random process with the continuous time t, - < t < -, then

1) $L^{+}(F_s) = L^{-}(F_s) = L(F_s)$;

2) $L^{+}(F_a) = L^{-}(F_a) = L(F_a)$ in that and only that case, if $\int\limits_{-\infty}^{\infty} \frac{|\ln f(\lambda)|}{1+\lambda^2}\, d\lambda = \infty$;

3) if

$$\int\limits_{-\infty}^{\infty} \frac{|\ln f(\lambda)|}{1+\lambda^2}\, d\lambda < \infty, \tag{2.2}$$

that f (λ) can be registered in the form $f(\lambda) = |g(\lambda)|^2$, where now g is an external function of class $\mathcal{H}^2$ in the upper half-plane Im z > 0, z = λ + iμ. Here

$$L^{-}(F_a) = D^{-} \cap L(F_a), \quad L^{+}(F_a) = D^{+} \cap L(F_a).$$

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Assertions 1) -3) both theorems are actually equivalent to those belonging A. N. Kolmogorov and M. G. Krein to the fundamental theorems of the theory of the forecast/prediction of stationary

random processes [1].

FOOTNOTE [1]. See, for example, [22]. ENDFOOTNOTE.

Since subsequently these assertions play fundamental role, we will give briefly their proofs.

First let us demonstrate paragraphs 1) -3) theorem 1. Let F be a singular measure on segment [$-\pi$, π]. Let us assume that $L^{-}\neq L$. Then it is necessary $e^{i\lambda}\notin L^{-}$. Let us designate by φ (λ) the projection of cell/element $e^{i\lambda}$ on subspace L^{-}. Then $e^{i\lambda}-\varphi(\lambda)\neq 0$ and $e^{i\lambda}-\varphi(\lambda)\perp L^{-}$, so that

$$\int_{-\pi}^{\pi} e^{in\lambda}(e^{i\lambda}-\varphi)F(d\lambda)=0,\ n=0,\ 1,\ \ldots$$

The generalized measure $F_1(d\lambda)=(e^{i\lambda}-\varphi)F(d\lambda)$ is analytical, i.e.,

$$\int_{-\pi}^{\pi} e^{in\lambda}F_1(d\lambda)=0,\quad n=0,\ 1,\ \ldots,$$

also, according to theorem ~~f.~~ F and M. Riesz ([10], p 73) must be absolutely continuous with Lebesgue. But this contradicts the singularity of measure F (dλ). The obtained contradiction proves p. 1).

Let us pass to p. 2). Let $\int_{-\pi}^{\pi}|\ln f(\lambda)|d\lambda=\infty$. Let us assume that in

spite of theorem $L^{-} \neq L$. Then again $e^{i\lambda} \notin L^{-}$, and if again ϕ (λ) there is a projection $e^{i\lambda}$ on L^{-}, then $\psi(\lambda) = e^{i\lambda} - \varphi(\lambda) \neq 0$ and

$$\int_{-\pi}^{\pi} e^{in\lambda}\psi(\lambda) f(\lambda)\, d\lambda = 0, \quad n = 0, 1, \ldots \qquad (2.3)$$

Let us designate $\|\varphi\|^{(p)}$ norm in space $\mathscr{L}^p(-\pi, \pi)$. Let us note that function $\psi f \in \mathscr{L}^1(-\pi, \pi)$. Actually, on Buniakowski's inequality $\|\psi f\|^{(1)} \leqslant \|\psi\|_{F_a} \cdot (\|f\|^{(1)})^{1/2} < \infty$.

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Hence on the basis of equality (2.3) it follows that $\psi f \in \mathscr{H}^1$. The logarithm of any function from $\mathscr{H}^1$ is summed, therefore,

$$\int_{-\pi}^{\pi} \ln|\psi(\lambda) f(\lambda)|\, d\lambda > -\infty.$$

From the elementary inequality ln x < x it follows that

$$\int_{-\pi}^{\pi} \ln|\psi^2(\lambda) f(\lambda)|\, d\lambda \leqslant \|\psi\|^2_{F_a} < \infty,$$
$$\int_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda \leqslant \int_{-\pi}^{\pi} f(\lambda)\, d\lambda < \infty.$$

Together with previous these inequalities mean that in spite of condition $\ln f \in \mathscr{L}^1$.

The obtained contradiction proves the first part p. 2), the second part p. 2) is contained in assertions p. 3), to proof of which we now will pass. On the strength of (2.1) f (λ) it is possible to register in the form $f(\lambda)=|g(e^{i\lambda})|^2$, where

$$g(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln|f(\theta)|d\theta\right\},\ z=re^{i\lambda},$$

is an external function from $\mathcal{H}^2$. Let $\varphi(e^{i\lambda})=\varphi\in L^+(F)$. This it indicates, then there exists the sequence of polynomials $P_n(z)$ such, that $\|\varphi-P_n\|_F\to 0$. But then and

$$\|\varphi g-P_n g\|^{(2)}=\|\varphi-P_n\|_F\to 0 \quad \text{npu}^{(1)} \quad n\to\infty.$$

Key: (1). with.

It is obvious, $P_n g\in\mathcal{H}^{2+}$. Therefore and limit function $\psi=\varphi g\in\mathcal{H}^{2+}$, i.e., $\varphi=\psi/g$, where $\psi,\ g\in\mathcal{H}^{2+}$. Being converted to canonical representation (1.4) of functions from $\mathcal{H}^{2+}$ and D^+, we see that $\varphi\in D^+$.

Conversely, let $\varphi\in D^+\cap L(F)$. Then $\psi=\varphi g\in\mathcal{H}^{2+}$. Function g - external, and according to the theorem of Byerling (see §1) the set of functions $\{gP\}$, where P passes whole polynomial, is dense in $\mathcal{H}^{2+}$. This that means specifically, that it is possible to find the sequence of polynomials P_n, for which with n → ∞

$$\|\psi - gP_n\|_2 = \|(\varphi - P_n)g\|_2 = \|\varphi - P_n\|_F \to 0,$$

i.e. that $\varphi \in L^+(F)$. The case $L^-(F)$ is examined analogously.

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The proof of paragraphs 1) -3) theorem 2 almost completely coincides with the proof of the corresponding pcint/items of theorem 1. Actually, with the help of the conformal mapping of circle onto half-plane it is proven, that the generalized theorem F. and M. Riesz is accurate in this case (see [20], p 209). Further, if $\varphi \in \mathscr{H}^2$ in the upper half-plane, then it is necessary

$$\int_{-\infty}^{\infty} \frac{|\ln|\varphi(\lambda)||}{1+\lambda^2}\,d\lambda < \infty.$$

Finally, in proof p. 3) instead of the theorem of Byerling it is to exile to Lax theorem.

It is useful to note that in proof p. 3) both theorems, actually, are obtained the following equalities: if are satisfied conditions (2.1) or (2.2), then

$$L^+ = \frac{1}{g}\mathscr{H}^{2+},\ L^- = \frac{1}{\bar{g}}\mathscr{H}^{2-}. \qquad (2.4)$$

§3. Structure of spaces $L_T(F)$, when T - the finite interval.

In previous §2 we traced the spaces $L_T(F)$ (but that means and spaces H (T), generated by the values of the corresponding stationary process ξ (t), $t \in T$, with the spectral measure F (dλ)), when T - infinite interval. Here there will be considered the case of the finite interval T = [a, b].

Since the case of discrete time is trivial: $L_T(F)$ consists of the trigonometric polynomials of form $\sum_{a \le t \le b} a_t e^{i\lambda t}$, below we will deal only with processes with continuous time. Furthermore, we will be bounded by study of the processes ξ (t), which have absolutely continuous spectral measure F (dλ) and the spectral density f (λ), that satisfies condition (2.2).

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As already mentioned above, sufficient to examine the intervals of form T = [-a, a]. Let us assume

$$L^{\sigma}(F) = \bigcap_{T=[-a,\, a],\ a>\sigma} L_T(F)$$

and

$$L^0(F) = \bigcap_T L_T(F).$$

The space $L^0(F)$ is determined by the behavior of process $\xi(t)$ in infinitesimal vicinity of zero; isometric to it the space $H^0 = \bigcap_T H_T$, in particular, contains all the existing derivatives $\xi^{(k)}(0)$.

Let us agree to designate through D_σ the set of the whole analytic functions $\phi(z)$, $z = \lambda + i\mu$, the final degree $\leqslant\sigma$, i.e., such whole functions, for which

$$\overline{\lim_{R\to\infty}} R^{-1} \max_\theta \ln|\varphi(Re^{i\theta})| \leqslant \sigma$$

(specifically, D_0 it designates the set of the integral functions of zero degree).

Theorem 3. If spectral density $f(\lambda)$ of stationary process $\xi(t)$ satisfies condition (2.2), then [1]

$$L^\sigma(F) = D_\sigma \dot{\cap} L(F), \quad L^0(F) = D_0 \cap L(F). \qquad (3.1)$$

FOOTNOTE [1]. We do not make the difference between functions from D_σ and their contraction to the real straight line $\mu = 0$.
ENDFOOTNOTE.

Proof. To us it is to demonstrate two connection/inclusions:

$$L^{\sigma}(F) \subset D_{\sigma} \cap L(F) \text{ and } L^{\sigma}(F) \supset D_{\sigma} \cap L(F).$$

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The first of them we will demonstrate for all $\sigma \geqslant 0$, the second - only for $\sigma = 0$ [1].

FOOTNOTE [1]. The proof of the general case ($\sigma > 0$) although is reached by the same means, substantially more cumbersome; reader can find this proof in the article of Levinson and Mckean (N. Levinson, H. McKean, Weighted trigonometrical approximation on R^1 with application to the germ field of a stationary gaussian noise, Acta Math. 112, No 1-2 (1964), 99-143). More powerful result is obtained previously M. G. Krein, who indicated integral representation for integral functions from $L^{\sigma}(F)$ (see "About the basic approximating problem of the theory of the extrapolation of stationary random processes", the DAN of USSR 94 (1954), 13-16).

The further results of this chapter are also borrowed from the cited article of Levinson and Mckean; their proof are somewhat changed. ENDFOOTNOTE.

1. Proof of connection/inclusion $\underbrace{L^{\sigma}(F) \subset D_{\sigma} \cap L(F).}$ Let function $\varphi \in L^{\sigma}(F)$. Will be located the functions

$$\varphi_n(\lambda) = \sum_j a_{jn} \exp\{it_{jn}\lambda\}, \quad |t_{jn}| \leqslant \sigma + 1/n,$$

such that $\underbrace{\|\varphi - \varphi_n\|_F < 1/n, \ n = 1, \ 2, \ \ldots}$ Is obvious, everything $\varphi_n \in D_{\sigma + 1/n}$.

We will demonstrate that at each point of the composite plane

$$|\varphi_n(z)| \leqslant C_{\varepsilon} e^{(\sigma + \varepsilon)|z|}, \qquad (3.2)$$

whatever the number $\varepsilon > 0$, the constants C_{ε} depending only on ε (but not of n).

The uniformly bounded family analytic functions φ_n is compact. Furthermore, $\|\varphi_n - \varphi\|_F \to 0$, and therefore $\varphi_n(z)$ it converge to the integral function ϕ (z), which under condition (3.2) there is a function of the final degree not larger σ. It is obvious, contraction ϕ (z) on Im z = 0 coincides with ϕ (λ).

Thus, remained to demonstrate inequality (3.2). For this we will estimate $|\varphi_n(z)|$ on the bisectrices of quardants and we will use Fragmena-Lindeloef's principle [2].

FOOTNOTE [2]. See, for example, [18]. ENDFOOTNOTE.

Let us estimate first $|\varphi_n(z)|$ for $|\mu| \geqslant 1$(

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Let us note that on the strength of (2.2) $f(\lambda) = |g(\lambda)|^2$, where g (z) - the external function of class $\mathscr{H}^2$. Let us introduce functions $\psi_n(z) = \varphi_n g \exp\{iz(\sigma+\delta)\},\ 0<\delta\leqslant\frac{1}{n}$. It is obvious, $\psi_n \in \mathscr{H}^2$, and therefore with all $\mu > 0$

$$\int_{-\infty}^{\infty} |\psi_n(\lambda+i\mu)|^2\,d\mu \leqslant \int_{-\infty}^{\infty} |\psi_n(\lambda)|^2\,d\lambda = \|\varphi_n\|_F^2 \leqslant$$
$$\leqslant \left(\|\varphi\|_F + \frac{1}{n}\right)^2 \leqslant (\|\varphi\|_F + 1)^2 = C_1. \quad (3.3)$$

Further, since functions $g \in \mathscr{H}^2$, it it is possible to present on the basis of the theorem of Peli - Weiner in the form

$$g(z) = \int_0^\infty e^{iuz}\hat{g}(u)\,du, \quad z = \lambda + i\mu, \quad \mu > 0,$$

where g (u) - Fourier transform to function g (λ). Therefore with all $\mu > 0$

$$|g(\lambda+i\mu)| \leqslant \left(\int_0^\infty |\hat{g}(u)|^2\,du\right)^{1/2}\left(\int_0^\infty e^{-2\mu u}\,du\right)^{1/2} = \frac{C_2}{\sqrt{\mu}}. \quad (3.4)$$

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GAUSSIAN RANDOM PROCESSES. PART 1,(U)

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Let us designate by Γ_R the outline, which consists of the segment of the direct/straight $|\lambda| \leqslant R,\ \mu = 1/2$ and being based on it circular arc of radius R with center of point z = i/2, Re z $\geqslant$ 1/2. By Cauchy formula, if Re $z_0 \geqslant 1/2$, and radius R is sufficiently great,

$$\psi_n(z_0) = \frac{1}{2\pi} \int\limits_{\Gamma_R} \frac{\psi_n(z)}{z - z_0}\, dz.$$

Relying on relationship/ratio (3.4), it is not difficult to show that the integral in terms of semicircumerence Γ_R of $\frac{\psi(z)}{z - z_0}$ approaches 0, when R $\to$ ∞. Consequently, withall $\mu \geqslant 1$ on the strength of inequalities (3.3)

$$|\psi_n(\lambda + i\mu)| \leqslant \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\left|\psi_n\left(u + \frac{i}{2}\right)\right|}{\left|(u - \lambda) + i\left(\frac{1}{2} - \mu\right)\right|}\, du \leqslant$$

$$\leqslant \frac{1}{2\pi}\left(\int\limits_{-\infty}^{\infty} \left|\psi_n\left(u + \frac{i}{2}\right)\right|^2 du \int\limits_{-\infty}^{\infty} \frac{du}{u^2 + \frac{1}{4}}\right)^{1/2} = C_3. \quad (3.5)$$

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Further, record/writing ln|φ (z)| in the form actually, with fixed/recorded T > 0 and z = Re

Further, record/writing $\ln|g(z)|$ in the form of the Poisson integral (g - external function, see 1):

$$\ln|g(z)| = \frac{\mu}{\pi} \int_{-\infty}^{\infty} \frac{\ln|g(u)|}{(u-\lambda)^2+\mu^2}\, du, \quad z = \lambda + i\mu,$$

we find that if $z = Re^{i\theta}$, $\theta = \frac{\pi}{4}$, $\frac{3\pi}{4}$, then when $R \to \infty$

$$\frac{\ln|g(z)|}{|z|} \to 0. \tag{3.6}$$

Actually, with fixed/recorded T > 0 and z = Re

$$\frac{\mu}{\pi} \int_{-T}^{T} \frac{\ln|g(u)|}{(u-\lambda)^2+\mu^2}\, du = O\left(\frac{1}{\mu}\right) = O\left(\frac{1}{R}\right) \to 0.$$

At the same time

$$\frac{\mu}{\pi}\left|\int\limits_{|u|>T}\frac{\ln|g(u)|}{(u-\lambda)^2+\mu^2}du\right|\leqslant\frac{\mu^2+\lambda^2+1}{2\pi\mu}\int\limits_{|u|>T}\frac{|\ln f(u)|}{1+u^2}du=$$
$$=\frac{R}{\pi\sqrt{2}}\frac{R^2+1}{R^2}\int\limits_{|u|>T}\frac{|\ln f(u)|}{1+u^2}du=R\cdot o(1),\quad T\to\infty.$$

Two last/latter relationship/ratios they prove (3.6).

From (3.6) it follows that for any $\varepsilon > 0$ and $z = Re^{i\theta}$ ($\theta = \pi/4$, $3\pi/4$; $R \geqslant 1$) is implemented inequality $|g(z)| \geqslant C_4 e^{-\varepsilon|z|}$, where constant C_4, possibly, depends on ε. Taking into account (3.5) let us have a hence on ray/beams $z = Re^{i\theta}$ ($\theta = \pi/4$, $3\pi/4$; $R \geqslant 1$) the following estimation:

$$|\varphi_n(z)|\leqslant C_5e^{(\sigma+\varepsilon+\delta)|z|}. \qquad (3.7)$$

It is analogous, introducing function $\psi_n^-(z) = \varphi_n\tilde{g}\exp\{-iz(\sigma+\delta)\}$, where for z from the lower half-plane $\tilde{g}(z) = \overline{g(\bar{z})} \in \mathcal{H}^{2-}$, we will obtain on ray/beams $z = Re^{i\theta}$, $\theta = 5\pi/4$, $7\pi/4$, $R \geqslant 1$ the estimation

$$|\varphi_n(z)|\leqslant C_5e^{(\sigma+\varepsilon+\delta)|z|}. \qquad (3.8)$$

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Let us now move on to the estimation $|\varphi_n(z)|$. on the segments of the bisectrices, which lie ~~those which lie~~ within circle $|z| \leqslant 1$.. As usual, let us count $\ln^+ a = \ln a$, if $a > 1$, and $\ln^+ a = 0$, if $a \leqslant 1$. Considering the subharmonic function $\ln |\varphi_n(z)|$, using the Poisson integral, we will have on straight lines

$$|\operatorname{Im} z| = 1$$

$$\begin{aligned}
\ln|\varphi_n(z)| &\leqslant \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln|\varphi_n(u)|}{(u-\lambda)^2+1} du \leqslant \\
&\leqslant \frac{1}{\pi} \sup_u \frac{u^2+1}{(u-\lambda)^2+1} \int_{-\infty}^{\infty} \frac{\ln^+|\varphi_n(u)|}{u^2+1} du \leqslant \\
&\leqslant \frac{1}{\pi} \frac{\lambda^2+2}{2} \int_{-\infty}^{\infty} \frac{\ln^+|\varphi_n(u)|}{u^2+1} du = \\
&= \frac{1}{\pi} \frac{\lambda^2+2}{2} \int_{-\infty}^{\infty} \frac{\ln^+ \frac{|\varphi_n(u)|\,|g(u)|}{|g(u)|}}{1+u^2} du \leqslant \\
&\leqslant \frac{1}{\pi} \frac{\lambda^2+2}{2} \left(\int_{-\infty}^{\infty} \frac{|\ln|\varphi_n(u)\,g(u)||}{1+u^2} du + \int_{-\infty}^{\infty} \frac{|\ln|g(u)||}{1+u^2} du \right) \leqslant \\
&\leqslant C_6(\lambda^2+2) \left(\|\varphi_n\|_F + \int_{-\infty}^{\infty} \frac{|\ln f(u)|}{1+u^2} du \right) \leqslant C_7(\lambda^2+2).
\end{aligned}$$

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Thus, function $e^{-C_7z^2}\varphi_n(z)$, analytical in band $|\mathrm{Im}\ z| \leqslant 1$, limited in this band, also satisfies on its boundaries equation $|e^{-C_7z^2}\varphi_n(z)| \leqslant C_8$, where C_8 does not depend on n. In accordance with the principle of Fragmena - Lindeloef the last/latter inequality it occurs in all points of the band indicated. Specifically, all the functions φ_n (z) are evenly limited in circle $|z| \leqslant 1$. Hence and from (3.7), (3.8) follows the validity of inequality (3.2), which, as already it was said, and proves 1st part of the theorem.

2. Proof of connection/inclusion L^0 (F) $\supset D_0 \cap$ L (F). Let ϕ (λ) $\in D_0 \cup$ L (F). According to Hadamard's known theorem [1] function ϕ (λ), whole function of zero degree, can be registered in the form of the product

$$\varphi(z) = z^m e^b \prod_1^\infty \left(1 - \frac{z}{z_n}\right) e^{z/z_n}, \quad \sum \frac{1}{|z_n|^2} < \infty, \qquad (3.9)$$

where $z_n \neq 0$ - zero ϕ (z).

FOOTNOTE [1]. See, for example, [18], page 525. ENDFOOTNOTE.

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Function ϕ (z) can be rewritten in the form of sum ϕ_1 (z) + ϕ_2 (z),

where $\varphi_1(z) = {}^1/_2(\varphi(z) + \varphi(-z))$ and $\varphi_2(z) = {}^1/_2(\varphi(z) - \varphi(-z))$ - with respect to even and odd function. With this as before $\phi_1, \phi_2 \in D_0 \cap L(F)$. Thus, to us it suffices to demonstrate theorem for even and odd functions. The proof in both these cases is identical, and we for a certainty will consider even functions $\phi(\lambda)$.

We should demonstrate that with any $\varepsilon > 0$ will be located the function $\varphi_\varepsilon \in L^\varepsilon(F)$, for which $\|\varphi - \varphi_\varepsilon\|_F \leqslant \varepsilon$.

Let us note first that any summarized with square integral function $\phi(\lambda)$ the final degree $\leqslant \varepsilon$ belongs $L^\varepsilon(F)$. Actually, any this function belongs to L (F), and according to the theorem of Peli - Weiner[1] relative to integral functions from $\mathscr{L}^2$ funcions ϕ has Pourier transform $\hat{\varphi}$, equal to zero outside $[-\varepsilon, \varepsilon]$, so that

$$\varphi(\lambda) = \int_{-\varepsilon}^{\varepsilon} e^{i\lambda u} \hat{\varphi}(u)\, du$$

and, of course, $\varphi \in L^\varepsilon(F)$.

FOOTNOTE [1]. See, for example, [10], page 82. ENDFOOTNOTE.

Thus, it suffices to construct the summarized with square whole function of degree $\leqslant \varepsilon$, which well approaches ϕ. Let us note that for

an even function ϕ the Adamarov of factorization (3.9) will take the form

$$\varphi(\lambda)=\lambda^{2m}\prod_{1}^{n}\left(1-\frac{\lambda^2}{z_n^2}\right),\quad m\geqslant 0.$$

Let us determine function $\varphi_\varepsilon(\lambda)$ by the equality

$$\varphi_\varepsilon(\lambda)=\lambda^{2m}\prod_{|z_n|<d}\left(1-\frac{\lambda^2}{z_n^2}\right)\prod_{n>d\delta}\left(1-\frac{\lambda^2\delta^2}{n^2}\right),$$

where $\delta=\frac{\varepsilon}{\pi}$, and number $d=d(\varepsilon)$ will be made more precise late.

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Let us show that φ_ε - the summarized with square integral function of degree $\leqslant\varepsilon$ (and, which means, $\varphi_\varepsilon\in L^2(F)$). The Euler formula

$$\sin\pi\lambda=\pi\lambda\prod_{1}^{\infty}\left(1-\frac{\lambda^2}{n^2}\right)$$

makes it possible to register φ_ε in the form

$$\varphi_\varepsilon(\lambda)=\frac{\sin\pi\delta\lambda}{\pi\delta}\,\frac{\lambda^{2m-1}\prod\limits_{|z_n|<d}\left(1-\frac{\lambda^2}{z_n^2}\right)}{\prod\limits_{n<d\delta}\left(1-\frac{\lambda^2\delta^2}{n^2}\right)},\qquad(3.10)$$

so that $\varphi_\varepsilon(\lambda)$ is the integral function of degree $\pi\delta=\varepsilon.$ Let us now move on to the estimation of the relation of polynomials in right side (3.10) with large λ. Let us introduce the monotonically

nondecreasing function $N_\varphi(R)$, to the number of roots of function ϕ (z) in circle $|z|\leqslant R$. Function $N_\varphi(R)$ closely related with the order of an increase in function ϕ (z). Specifically, for the functions of zero degree $N_\varphi(R)=o(R),\ R\to\infty$ [1].

FOOTNOTE [1]. See [18], p 521. ENDFOOTNOTE.

Consequently, degree of polynomial in the numerator of expression (3.10) is o (d), whereas degree of polynomial in denominator ~2dδ, and $\varphi_\varepsilon\in\mathscr{L}_2(-\infty,\infty)$ for sufficiently large d.

To us remained to consider $\|\varphi_\varepsilon-\varphi\|_F$. For this let us demonstrate the following lemma, which characterizes proximity φ_ε to ϕ.

Problem 1. At assigned $\varepsilon>0,\ A<\infty$ it is possible to find number d_0 such that for all d > d_0 they occur of the inequality

$$\max_{|\lambda|\leqslant A}|\varphi(\lambda)-\varphi_\varepsilon(\lambda)|\leqslant\frac{\varepsilon}{\sqrt{2}}\left(\int_{-A}^{A}f(\lambda)\,d\lambda\right)^{-1/2},$$
$$\max_{A\leqslant|\lambda|\leqslant\frac{d}{2}}\left|\frac{\varphi_\varepsilon(\lambda)}{\varphi(\lambda)}\right|\leqslant1,\quad\max_{|\lambda|\geqslant\frac{d}{2}}|\varphi(\lambda)|\leqslant1.\tag{3.11}$$

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The first of inequalities (3.11) is obvious. For the proof of the second we will use the elementary inequality

$$e^{-2x} \leqslant 1 - x \leqslant e^{-x}, \quad 0 \leqslant x < \frac{1}{4},$$

and let us write

$$\left|\frac{\varphi_\varepsilon}{\varphi}\right| = \frac{\prod\limits_{n > d\delta}\left(1 - \frac{\lambda^2\delta^2}{n^2}\right)}{\prod\limits_{|z_n| \geqslant d}\left|1 - \frac{\lambda^2}{z_n^2}\right|} \leqslant \frac{\prod\limits_{n > d\delta}\left|1 - \frac{\lambda^2\delta^2}{n^2}\right|}{\prod\limits_{|z_n| \geqslant d}\left(1 - \frac{\lambda^2}{|z_n|^2}\right)} \leqslant$$

$$\leqslant \exp\left\{-\lambda^2\delta^2 \sum_{n > d\delta} \frac{1}{n^2}\right\} \exp\left\{2\lambda^2 \sum_{|z_n| \geqslant d} \frac{1}{|z_n|^2}\right\}. \quad (3.12)$$

To evaluate the first factor in (3.12) let us note that

$$\sum_{n > d\delta} \frac{1}{n^2} \geqslant \int_{d\delta+1} \frac{dx}{x^2} = \frac{1}{d\delta + 1}.$$

In order to consider $\sum\limits_{|z_n| \geqslant d} \frac{1}{|z_n|^2}$, let us turn again to distribution function of zeros $N_\varphi(R)$. Taking into account that $N_\varphi(R) = o(R)$, let us have

$$\sum_{|z_n| \geqslant d} \frac{1}{|z_n|^2} = \int_d^\infty \frac{dN_\varphi(R)}{R^2} = -\frac{N_\varphi(d)}{d^2} + 2\int_d^\infty \frac{N_\varphi(R)}{R^3}\,dR \leqslant$$

$$\leqslant 2 \sup_{R \geqslant d} \frac{N_\varphi(R)}{R} \int_d^\infty \frac{dR}{R^2} = \frac{2}{d} \sup_{R \geqslant d} \frac{N_\varphi(R)}{R} \leqslant \frac{\delta}{2d},$$

if only d selected by sufficiently large. after substituting the obtained estimations in (3.12), we will obtain that

$$\max_{|\lambda|\leqslant d/2}\left|\frac{\varphi_\varepsilon(\lambda)}{\varphi(\lambda)}\right|\leqslant \max_{|\lambda|\leqslant d/2}\exp\left\{-\lambda^2\left(\frac{\delta^2}{d\delta+1}-\frac{\delta}{2d}\right)\right\}=1,$$

as soon D $\geqslant$ 1/δ. Second inequality (3.11) is proved.

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Let us pass to the proof of the last/latter inequality of lemma. Now $|\lambda|\geqslant d/2$. By considering the second of the factors in expression for φ_ε, let us find that

$$\ln\left(\prod_{|z_n|<d}\left|1-\frac{\lambda^2}{z_n^2}\right|\right)=\sum_{|z_n|<d}\ln\left|1-\frac{\lambda^2}{z_n^2}\right|=$$

$$=\int_0^d\ln\left|1-\frac{\lambda^2}{R^2}\right|dN_\varphi(R)\leqslant\int_0^d\ln\left(1+\frac{\lambda^2}{R^2}\right)dN_\varphi(R)=$$

$$=N_\varphi(d)\ln\left(1+\frac{\lambda^2}{d^2}\right)+\int_0^d\frac{2\lambda^2}{\lambda^2+R^2}\frac{N_\varphi(R)}{R}dR\leqslant$$

$$\leqslant 2N(d)\ln\frac{5|\lambda|}{d}+2\int_0^d\frac{N_\varphi(R)}{R}dR=$$

$$=o(d\ln d)+o\left(d\ln\frac{5|\lambda|}{d}\right).\quad(3.13)$$

To evaluate the last/latter factor in formula for φ_e let us consider separately cases $d/2 \leqslant |\lambda| \leqslant d$ and $|\lambda| > d$. If $\lambda \leqslant d$, then

$$\prod_{n > d\delta}\left(1 - \frac{\lambda^2\delta^2}{n^2}\right) \leqslant \exp\left\{-\lambda^2\delta^2 \sum_{n > d\delta} \frac{1}{n^2}\right\} \leqslant$$
$$\leqslant \exp\left\{-\frac{\lambda^2\delta^2}{d\delta + 1}\right\} \leqslant e^{-\frac{\lambda\delta}{2}},$$

provided $d\delta \geqslant 1$. If $|\lambda| > d$, then, again by using the expansion of Euler, let us find that

$$\prod_{n > d\delta}\left|1 - \frac{\lambda^2\delta^2}{n^2}\right| = \frac{\sin \pi\delta\lambda}{\pi\delta\lambda \prod\limits_{n < d\delta}\left|1 - \frac{\lambda^2\delta^2}{n^2}\right|}.$$

For $n < d\delta/\sqrt{2}$ will be $\left|1 \; \lambda^2\delta^2/n^2\right| > \lambda^2\delta^2/2n^2$, so that on the basis of the Stirling formula for n! let us have

$$\prod_{n>d\delta}\left|1-\frac{\lambda^2\delta^2}{n^2}\right|\leqslant\frac{1}{\pi\delta}\exp\left\{-d\delta\sqrt{2}\ln\frac{|\lambda|\delta}{\sqrt{2}}+\right.$$
$$\left.+d\delta\sqrt{2}\ln\frac{d\delta}{\sqrt{2}}\right\}=\frac{1}{\pi\delta}\exp\left\{-d\delta\sqrt{2}\ln\frac{|\lambda|}{d}\right\}.$$

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Equate/comparing the obtained estimations with (3.13), we see that as soon as d is sufficiently great, a $|\lambda|\geqslant d/2$, then $|\varphi_\varepsilon(\lambda)|\leqslant 1.$ Lemma is demonstrated.

With the help of this lemma immediately is obtained the estimation for $\|\varphi-\varphi_\varepsilon\|_F$. Specifically, on the strength of (3.11)

$$\|\varphi-\varphi_\varepsilon\|_F^2\leqslant\int_{-A}^{A}|\varphi(\lambda)-\varphi_\varepsilon(\lambda)|^2 f(\lambda)\,d\lambda+$$
$$+5\int_{|\lambda|\geqslant A}|\varphi(\lambda)|^2 f(\lambda)\,d\lambda+2\int_{|\lambda|>d/2}f(\lambda)\,d\lambda<\varepsilon^2,$$

if A and d are selected by sufficiently large. Theorem 3 is demonstrated.

§4. Projection $L^+(F)$ on $L^-(F)$.

It is well known [1], how important role in the theory of forecast/prediction play the projections of spaces $L_T^-(F)$, $T=[a,\infty)$, $a\geqslant 0$, on space $L^-(F)$.

FOOTNOTE [1]. See, for example, [22]. ENDFOOTNOTE.

These projections appear during the study of the varied conditions of the regularity of random processes (see below chapter IV). Here we will demonstrate a series of the theorems about the structure of space $L^{+|-}$ (F) - to projection $L^+(F)$ on $L^-(F)$.

On the strength of theorems 1, 2, it suffices to examine the random processes, which possess spectral density f (λ), which satisfies conditions (2.1) (for processes with discrete time) or (2.2) (for processes with continuous time). Recall that in that case spectral density f (λ) allow/assumes the factorization

$$f(\lambda)=|g(e^{i\lambda})|^2=g(e^{i\lambda})\cdot\bar{g}(e^{i\lambda})$$

(time t discretely), where g (z) - the function of class $\mathcal{H}^{2+}$, $|z|<1$, a $\bar{g}(z)=\overline{g\left(\frac{1}{\bar{z}}\right)}$, $|z|>1$-the function of class $\mathcal{H}^{2-}$. It is analogous for processes with the continuous time

$$f(\lambda)=|g(\lambda)|^2=g(\lambda)\cdot\bar{g}(\lambda),$$
$$g\in\mathcal{H}^{2+},\quad \operatorname{Im}z>0;\quad \bar{g}(z)=\overline{g(\bar{z})},\quad \operatorname{Im}z<0,\quad \bar{g}\in\mathcal{H}^{2-}.$$

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Are obvious the connection/inclusions

$$L^-(F) \supseteq L^{+|-}(F) \supseteq L^-(F) \cap L^+(F) \supseteq L^{\circ}(F). \quad (4.1)$$

The basic our target/purpose - to explain, when these or other of these inclusions are transformed into equalities.

We will begin from the description of the analytical form of the operators $\mathscr{P}^-$ (designated also $\mathscr{P}$) and $\mathscr{P}^+$, design/projecting $L_\pi(F)$ respectively for $L^-(F)$ and $L^+(F)$. Let us designate by π^- $(=\pi)$ and π^+ the operators are projected on $\mathscr{L}^2(-\infty, \infty)$ $(\mathscr{L}^2(\pi, \pi)$ on $\mathscr{H}^{2-}$ and $\mathscr{H}^{2+}$ respectively. These operators are determined as follows: let $\phi \in \mathscr{L}^2$ $(-\infty, \infty)$ and

$$\varphi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda u}\hat{\varphi}(u)\,du, \text{ where: } \hat{\varphi} \in \mathscr{L}^2(-\infty, \infty);$$

then

$$\pi^-\varphi(\lambda) = \int_{-\infty}^{0} e^{i\lambda u}\hat{\varphi}(u)\,du, \quad \pi^+\varphi(\lambda) = \int_{0}^{\infty} e^{i\lambda u}\varphi(u)\,du.$$

It is analogous, if $\varphi \in \mathscr{L}^2(-\pi, \pi), \varphi(\lambda) = \sum_{-\infty}^{\infty} e^{i\lambda l}a_l$, where

then

$\sum |a_l|^2 < \infty$, or

$$\pi^-\varphi = \sum_{-\infty}^{0} e^{i\lambda l}a_l, \qquad \pi^+\varphi = \sum_{0}^{\infty} e^{i\lambda l}a_l.$$

Lemma 2. Occur the following relationship/ratios:

$$\mathscr{P}^{-}\varphi(\lambda)=\bar{g}^{-1}(\lambda)\pi^{-}\bar{g}(\lambda)\varphi(\lambda), \quad \mathscr{P}^{+}=g^{-1}\pi^{+}g.$$

Proof. We will be bounded to examination $\mathscr{P}^{-}=\mathscr{P}$. Let us also assume that we deal with the continuous time t (case of discrete time is examined analogously). We should demonstrate that

$$\mathscr{P}^{*}=\mathscr{P},\ \mathscr{P}^{2}=\mathscr{P},\ \mathscr{P}L(F)\subseteq L^{-}(F),\ \ \mathscr{P}L^{-}(F)=L^{-}(F).$$

two first equalities mean that $\mathscr{P}$ - projector [1], two following, that $\mathscr{P}$-projector on $L^{-}(F)$.

FOOTNOTE [1]. See, for example, [2], page 111. ENDFOOTNOTE.

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Since π - projector, $\pi^{*}=\pi$, $\pi^{2}=\pi$. Therefore

$$\begin{aligned}\langle\mathscr{P}\varphi,\psi\rangle_{F}&=(\pi(\bar{g}\varphi),\ \bar{g}\psi)_{2}=(\bar{g}\varphi,\ \pi(\bar{g}\psi))_{2}=\\&=\left(g\bar{g}\varphi,\ \bar{g}^{-1}\pi\bar{g}\psi\right)_{2}=\langle\varphi,\ \mathscr{P}\psi\rangle_{F}.\end{aligned}$$

Equality $\mathscr{P}^{2}=\mathscr{P}$ with obviousness follows from relationship/ratio $\pi^{2}=\pi$.

Further, in §2 it was shown, that if $\varphi\in L^{-}(F)$, then $\bar{g}\varphi\in\mathscr{H}^{2-}$, if $\psi\in\mathscr{H}^{2-}$, then $\psi\bar{g}^{-1}\in L^{-}(F)$. Since $\pi(\bar{g}\varphi)\in\mathscr{H}^{2-}$, then $\mathscr{P}\varphi=\bar{g}^{-1}\pi\bar{g}\varphi\in L^{-}(F)$. If finally $\varphi\in L^{-}(F)$, then

$$\mathscr{P}\varphi=\bar{g}^{-1}\pi(\bar{g}\varphi)=\bar{g}^{-1}\bar{g}\varphi=\varphi.$$

Lemma is demonstrated.

Theorem 4. $L^{-}(F) = L^{+|-}(F)$ in that and only that case, if $\theta = g/\bar{g}$ coincides with the relation of two internal functions; $L^{+|-}(F) = L^{-}(F) \cap L^{+}(F)$ in that and only that case, if $\theta = g/\bar{g}$ there is an internal function.

Proof. Theorem is proven by similar form as for processes with discrete, so also with continuous time. We will give the proof of the first part of the theorem for discrete time and the second - for continuous.

Let $\theta = s_1(e^{i\lambda})/s_2(e^{i\lambda})$, where $s_1(z)$, $s_2(z)$ are internal functions, so that $|s_1(e^{i\lambda})| = |s_2(e^{i\lambda})| \equiv 1$. Let us consider that $s_1(0) = s_2(0) = 0$; otherwise we could, without varying θ, replace s_1, s_1 by zs_1, zs_2. Let us consider function $\phi = \bar{g}^{-1}(s_1\bar{g})$ and show that $\varphi \in L^{-}(F)$, but $\varphi \perp L^{+|-}(F)$(it goes without saying that $\phi = 0$), so that $L^{-} \neq L^{+|-}$. It is obvious $\bar{s}_1\bar{g} \in \mathcal{H}^{2-}$ and therefore $\varphi \in L^{-}$. Further, ϕ it is possible to register in the form

$$\varphi = \frac{1}{g}\frac{g}{\bar{g}}(\bar{s}_1\bar{g}) = \frac{1}{g}\frac{|s_1|^2}{s_2}\bar{g} = \frac{1}{g}(\bar{s}_2\bar{g}),$$

and, since $s_2(0) = 0$, $\varphi g \in e^{-i\lambda}\mathcal{H}^{2-}$. Let now $\psi-$ be the arbitrary

cell/element $L^+(F)$. We have

$$\langle \varphi, \mathscr{P}\psi\rangle_F = (\varphi\bar{g}, \pi(\bar{g}\psi))_2 = \\ = (\pi(\varphi\bar{g}), \bar{g}\psi)_2 = (\varphi\bar{g}, \bar{g}\psi)_2 = (\varphi g, \psi g)_2 = 0,$$

since

$$\varphi g \in e^{-i\lambda}\mathscr{H}^{2-}, \ \psi g \in \mathscr{H}^{2+}.$$

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Conversely, let $L^- = L^{+|-}$ and let $\varphi \in L^-$, but $\varphi \perp L^{+|-}$. Let us register function φ in the form $\bar{\varphi}_1/\bar{g}$, $\varphi_1 \in \mathscr{H}^{2+}$. Function $\varphi_1 \in \mathscr{H}^2$ can be registered in the form of product $\varphi_1 = \theta_1\gamma_1$, where θ_1 - internal function, a $\gamma_1 \in \mathscr{H}^2$ - is external (see §1). Let us note now that the function φg is orthogonal $\mathscr{H}^{2+}$ in $\mathscr{L}^2(-\pi, \pi)$. Actually, if $\psi \in \mathscr{H}^{2+}$, then

$$(\varphi g, \psi)_2 = \left(\varphi\bar{g}, \frac{\bar{g}}{g}\psi\right)_2 = \left(\pi(\varphi\bar{g}), \frac{\bar{g}}{g}\psi\right)_2 = \\ = \left(\frac{1}{\bar{g}}\pi\bar{g}\varphi, \frac{\psi}{g}|g|^2\right)_2 = \left\langle \mathscr{P}\varphi, \frac{\psi}{g}\right\rangle_F = \left\langle \varphi, \mathscr{P}\frac{\psi}{g}\right\rangle_F = 0,$$

then since $\psi/g \in L^+(F)$. Therefore $\varphi g \in \mathscr{H}^{2-}$, and, which means, $\varphi g = \bar{\theta}_2 \cdot \bar{\gamma}_2$, where θ_2, γ_2 - with respect to internal and external function. Thus, we have

$$\varphi g = \frac{g}{\bar{g}}\bar{\varphi}_1 = \frac{g}{\bar{g}}\bar{\theta}_1\bar{\gamma}_1 = \bar{\theta}_2\bar{\gamma}_2.$$

Since $|\theta_1| = |\theta_2| = 1$ (θ_1, θ_2 - internal), $|\gamma_1| = |\gamma_2|$. External functions $\gamma_1(z)$, $\gamma_2(z)$ whose module/moduli to $|z| = 1$ coincide, are equal, so that

$$\frac{g}{\bar{g}} = \frac{\bar{\theta}_2}{\bar{\theta}_1} = \frac{\theta_1}{\theta_2}.$$

The first part of the theorem is demonstrated.

The proof of 2-1 parts of the theorem. Let us note first that in accordance with the done previously assumptions $L^{-}(F) \neq L(F)$ and, which means, $L^{-}(F) \neq L^{-}(F) \cap L^{+}(F)$. Therefore, according to that which was demonstrated above, during the investigation of the question concerning equality $L^{-} \cap L^{+} = L^{+|-}$ it is possible to count $\theta = \theta_1/\theta_2$ where θ_1, θ_2 are internal functions without common divisor.

Proof is based on the following equality:

$$L^{+|-}(F) = L^{-}(F) \cap \frac{1}{\theta_2} L^{+}(F), \qquad (4.2)$$

from which immediately it follows that $L^{+|-} = L^{-} \cap L^{+}$ when and only when $\theta_2 \equiv 1$, i.e., when $\theta = \theta_1/\theta_2 \equiv \theta_1$ is an internal function.

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For proof (4.2) let us introduce into examination orthogonal complement M of space $L^{+|-}$ to L^{-} and show that

$$M = L^{-} \cap \frac{\bar{g}}{g} L^{-} = \frac{1}{\bar{g}} \mathscr{H}^{2-} \cap \frac{1}{g} \mathscr{H}^{2-}. \qquad (4.3)$$

Actually, let us register arbitrary cell/element $\varphi \in M \subset L^{-}$ in the form $\varphi = \bar{\varphi}_1 \sqrt{\bar{g}}$, where $\varphi_1 \in \mathscr{H}^2$. If φ is orthogonal $L^{+|-} = \mathscr{P} L^{+} = \frac{1}{\bar{g}} \pi \frac{\bar{g}}{g} \mathscr{H}^{2+}$, then $\bar{\varphi}_1$ is orthogonal in $\mathscr{L}^2(-\infty, \infty)$ to space $\pi \frac{\bar{g}}{g} \mathscr{H}^{2+}$. But

since $\bar{w}\bar{\Phi}_1 = \bar{\Phi}_1$, this means that $\bar{\Phi}_1$ is orthogonal $\frac{\bar{g}}{g}\mathscr{H}^{2+}$, a $\frac{g}{\bar{g}}\bar{\varphi}_1 = \varphi \cdot g$ is orthogonal to entire $\mathscr{H}^{2+}$. Consequently, $\varphi g \in \mathscr{H}^{2-}$, which is equivalent (4.3).

Is decomposed now function $\bar{\Phi}_1$ (belonging on the strength of equality (4.3) both $\mathscr{H}^{2-}$, and $\theta^{-1}\mathscr{H}^{2-}$) into the product of internal and exteriors. We will have

$$\bar{\varphi}_1 = \bar{\gamma}_3\bar{\theta}_3 = \frac{\theta_2}{\theta_1}\bar{\gamma}_4\bar{\theta}_4,$$

where γ_i - external, a θ_i - internal factors. It is obvious, $\gamma_3 = \gamma_4$, so that

$$\theta_1\theta_4 = \theta_2\theta_3,$$

and, since θ_2 and θ_1 do not have common faactors, function θ_2 must be divided θ_4. Consequently, (4.3) it is possible to rewrite thus:

$$M = \frac{1}{\bar{g}}\mathscr{H}^{2-} \cap \frac{1}{g\theta_2}\mathscr{H}^{-2} = L^- \cap \frac{\bar{g}}{g\theta_2}L^-. \qquad (4.4)$$

Now, relying on relationship/ratio (4.4), let us find orthogonal complement M to L^- (equal regarding $L^{+|-}$). Let us note that

$$\frac{\bar{g}}{g}\frac{1}{\theta_2}L^- = \theta_1 L^- = \frac{1}{\bar{g}}\theta_1\mathscr{H}^{-2} \subset \frac{1}{\bar{g}}\mathscr{H}^{2-} = L^-.$$

Therefore is sufficient to find orthogonal complement $\frac{\bar{g}}{g}\frac{1}{\theta_2}L^-$ to entire L (F) or, which is the same thing, to find orthogonal complement in $\mathscr{L}^2(-\infty, \infty)$ spaces $\frac{1}{\theta_2}\mathscr{H}^{2-}$ to in all $\mathscr{L}^2(-\infty, \infty)$.

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The latter, obviously, is $\frac{1}{\theta_2}\mathscr{H}^{2+}$, what in L (F) it answers space $\frac{1}{\theta_2}L^+(F)$. Equality (4.2), and with it and theorem are demonstrated.

We will explain now, when $L^{+|-}$ (F) = L^0 (F) (it goes without saying that already the very posing of the question assumes that we deal with process with continuous time). In light of theorem 3 in the case of the equality indicated there is full/total/complete description of space $L^{+|-}$.

Theorem 5. Let be is satisfied condition (2.2). Equality $L^{+|-}$ (F) = L^0 (F) occurs in that and only that case, if 1/f (λ) there is the whole analytic function of zero degree.

Proof. Let $L^{+|-}$ (F) = L^0 (F). Let us consider separately of two following cases.

1. $f/(1+\lambda^2)=f_1\in\mathscr{L}^1(-\infty,\infty)$. Let us introduce into examination new spectral measure with spectral density f_1. It is obvious, $f_1 = |g_1|^2$, where $g_1 = g/(\lambda + i) \in \mathscr{H}^{2+}$. We will demonstrate that $1/\bar{g}\in L^0(F_1)$. Hence and from theorem 3 it will follow that 1/g, and also, therefore, $1/f = 1/g\bar{g}$ essence the integral functions of zero degree.

On the basis (4.1) occurs equality $L^{+|-}(F) = L^{-}(F) \cap L^{+}(F)$, and according to theorem 4 $g/\bar{g} = \theta$ is an internal function. It is understandable that in that case $\theta_1 = g_1/\bar{g}_1 = \theta\ \lambda-i/\lambda+i$ - also internal function. Let us consider function $\varphi = \frac{\theta}{g_1(\lambda+i)} \in L^{+}(F_1)$ and find its projection on $L^{-}(F_1)$. We have, considering that here $\mathscr{P}$ - the projector in $L(F_1)$ on $L^{-}(F_1)$,

$$\mathscr{P}\varphi = \frac{1}{\bar{g}_1}\pi\bar{g}_1\varphi = \frac{1}{\bar{g}_1}\pi\frac{\bar{g}_1}{g_1}\frac{\theta}{\lambda+i} = \\ = \frac{1}{\bar{g}_1}\pi\frac{\lambda+i}{\lambda-i}\frac{1}{\theta_1}\frac{\theta_1}{\lambda+i} = \frac{1}{\bar{g}_1(\lambda-i)} = \frac{1}{\bar{g}},$$

since $\frac{1}{\lambda-i} = i\int_{-\infty}^{0} e^{iu(\lambda-i)}du$ belongs $\mathscr{H}^{2-}$ and $\pi\frac{1}{\lambda-i} = \frac{1}{\lambda-i}$.

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Thus, we demonstrated that $1/\bar{g} = \mathscr{P}\varphi \in L^{+|-}(F_1) = \dot{L}^{0}(F_1)$.

2. $f_1 \notin \mathscr{L}'(-\infty, \infty)$. In this case (see §3) space $L^{0}(F)$, and also, therefore, equal to it $L^{+|-}(F)$ contain only the constants, dim $L^{+|-}$ = 1. Of proven below theorem 6 it follows that is necessary $f = const/\ 1 + \lambda^2$.

The first part of the theorem is demonstrated. Let us begin toward the proof of 2-1 parts. Let $1/f(\lambda) = b(\lambda)$ there is the integral function of kind zero. Let us demonstrate first that $L^{+|-}$ =

$L^+ \cap L^-$. In accordance with theorem 4 for this it suffices to show that the function $\theta = g/\bar{g}$ is internal.

According to one theorem of N. I. Akhiyezer (See [16], of page 567) the nonnegative on real axis function b (λ) we will represent in the form

$$b(\lambda) = \omega(\lambda)\,\bar{\omega}(\lambda) = |\omega(\lambda)|^2,$$

where ω (z) and $\bar{\omega}$ (z) - the integral functions of kind zero, function ω are the external function in the upper half-plane, which does not have there as zeros, and $\bar{\omega}(z) = \overline{\omega(\bar{z})}$ - is external in lower half-plane. But then $g = 1/\omega$, $\bar{g} = 1/\bar{\omega}$. Let, further, z_l - zero function b (z), that lie at Inz < 0. For Im z > 0 $g/\bar{g} = \bar{\omega}/\omega$ are analytic function, whereupon

$$|\theta(z)| = \left|\frac{g(z)}{\bar{g}(z)}\right| = \left|\frac{\bar{\omega}(z)}{\omega(z)}\right| = \prod_n \left|\frac{1 - \frac{z}{z_n}}{1 - \frac{z}{\bar{z}_n}}\right| \leqslant 1, \quad |\theta(\lambda)| = 1.$$

The last/latter relationship/ratios prove, what θ - is internal.

Now us it remained to show that each cell/element $\varphi \in L^+ \cap L^-$ are the integral function of kind zero. Before let us demonstrate that ϕ - integral function. On the strength of (2.4)

$$\varphi(\lambda) = \omega(\lambda)\,h^+(\lambda) = \bar{\omega}(\lambda)\,h^-(\lambda),$$

where $h^+ \in \mathscr{H}^{2+}$, $h^- \in \mathscr{H}^{2-}$. Consequently, ϕ (λ) there is the common

boundary value of functions ω (z) h^{+}(z) and $\bar{\omega}$ (z) h^{-} (z), analytical respectively in IM z > 0 and Im z < 0.

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Let us demonstrate the possibility of their continuation through the real straight line.

Let $\Phi(z)=\int\limits_0^z \varphi(\xi)\,d\xi,$ where the integral is taken on the cutting off of straight line, that combines points 0, z, a ϕ (ξ) is either ω (ξ) h^{+}(ξ), or $\bar{\omega}$ (ξ) h^{-} (ξ). Function Φ is analytical in the upper and lower half-planes.

For function $h^{+}\in\mathcal{H}^{2+}$

$$\int\limits_{-\infty}^{\infty}|h^{+}(\lambda+i\mu)|^2\,d\lambda\leqslant\int\limits_{-\infty}^{\infty}|h^{+}(\lambda)|^2\,d\lambda=C_1<\infty,\quad \mu>0. \quad (4.5)$$

Therefore with all a, |a| < ∞,

$$\int\limits_0^a|\varphi(\lambda+i\mu)|\,d\lambda=\int\limits_0^a|\omega(\lambda+i\mu)\,h^{+}(\lambda+i\mu)|\,d\mu\leqslant$$
$$\leqslant\max_{|z|\leqslant a}|\omega(z)|\sqrt{a}\,C_1^{1/2}\leqslant C_2e^{a}. \quad (4.6)$$

According to the theorem of Peli - Weiner

$$h^+(z) = \int_0^\infty e^{izu} h(u)\,du,$$
$$h \in \mathscr{L}^2(-\infty, \infty), \quad \operatorname{Im} z > 0,$$

so that

$$|h^+(\lambda + i\mu)| \leqslant \left(\int_0^\infty e^{-2\mu u}\,du\right)^{1/2} C_1^{1/2} \leqslant \sqrt{\frac{C_1}{2\mu}}. \quad (4.7)$$

Therefore

$$\int_0^\varepsilon |\varphi(a + i\mu\,|\,d\mu = \int_0^\varepsilon |\omega(a + i\mu) h^+(a + i\mu)|\,d\mu \leqslant C_3 e^a \sqrt{\varepsilon}. \quad (4.8)$$

For almost all λ

$$\lim_{\mu \to 0} h^+(\lambda + i\mu) = h^+(\lambda).$$

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Hence and from (4.6) it follows that and

$$\lim_{\mu \to 0} \int_{i\mu}^{\lambda + i\mu} \varphi(\xi)\,d\xi = \int_0^\lambda \varphi(\xi)\,d\xi. \quad (4.9)$$

Integral of analytic function ϕ (ξ) in terms of the sides of triangle with apex/vertexes at points 0, $z = \lambda + i\mu$, λ is equal to zero. On the strength of (4.8) and (4.9)

$$\lim_{\mu \to 0} \Phi(z) = \lim_{\mu \to 0} \int_0^z \varphi(\xi)\,d\xi = \int_0^\lambda \varphi(\xi)\,d\xi = \Phi(\lambda), \quad \operatorname{Im} z > 0.$$

Analogous result is accurate also for z from lower half-plane. Consequently, function $\Phi(z)$ is analytical in the upper and lower half-planes and is continuous in an entire composite plane. Converse theorem the Cauchy-theorem of Morer (see [18], of page 186) they will make it possible to claim that $\Phi(z)$ is analytical in an entire plane. But then derivative $\varphi(z)=\Phi(z)$ also there is integral function.

To us it remained to consider the rate of growth $|\varphi(Re^{i\theta})|, R\to\infty$. On circumference $|z|=R$ $|\omega(z)|\leqslant C_\varepsilon e^{\varepsilon R}$, $|\tilde{\omega}(z)|\leqslant C_\varepsilon e^{\varepsilon R}$. As concerns functions $h^+(z)$, $h^-(z)$, on Peli-Weiner's theorem for $z=Re^{i\theta}$, $0<\theta<\pi$,

$$|h^+(z)|=\left|\int_0^\infty e^{izu}h(u)\,du\right|\leqslant\int_0^\infty e^{-R\sin\theta u}|h(u)|\,du\leqslant$$
$$\leqslant\frac{1}{\sqrt{2R\sin\theta}}\cdot\left(\int_0^\infty|h(u)|^2\,du\right)^{1/2}=\sqrt{\frac{C_1}{2R\sin\theta}}.$$

It is analogous on $z=Re^{i\theta}$, $\pi<\theta<2\pi$,

$$|h^-(z)|\leqslant\sqrt{\frac{C_1}{2R\,|\sin\theta|}}.$$

Therefore with large R

$$\int_0^{2\pi}\ln^+|\varphi(Re^{i\theta})|\,d\theta\leqslant\varepsilon R. \qquad (4.10)$$

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On the basis of the formula Poisson-of Jensen (see [18], of page 456)

$$\ln|\varphi(Re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} \ln|\varphi(\rho e^{i\alpha})| \frac{\rho^2 - R^2}{\rho^2 + R^2 - 2R\rho\cos(\alpha - \theta)} d\alpha + \\ + \sum \ln \frac{\rho|z - a_u|}{|\rho^2 - \bar{a}_u z|} - \alpha_0 \ln \frac{\varepsilon\rho}{R},$$

where $z = Re^{i\theta}$, $\rho > R$, sum is common to the right for all the zero $a_u \neq 0$ functions ϕ (z), which lie at circle $|z| \leqslant \rho$, and α_0, - multiplicity of zero ϕ (z) at point z = 0. If we assume here ρ = 2R and to note that ln $\frac{\rho|z - a_u|}{|\rho^2 - \bar{a}_u z|} \leqslant 0$, we will have on the strength of (4.10)

$$\ln|\varphi(Re^{i\theta})| \leqslant \frac{3}{2\pi}\int_0^{2\pi} \ln^+|\varphi(2Re^{i\alpha})|\, d\alpha \leqslant \varepsilon R.$$

Theorem is demonstrated.

Theorem 6. Let the spectral measure F be absolutely continuous. Space $L^{+|-}$ (F) has the final dimensionality n in that and only that case, if spectral density be rational integral function λ degree $2n$ for processes with continuous time or rational integral function degree $2n$ for processes with discrete time.

Proof. 1. ξ (t) - process with continuous time. Let dim $L^{+|-}$ (F) = n < ∞. Let us designate $\varphi_1, \ldots, \varphi_n$ any base in $L^{+|-}$ (F), and let $\eta_1, \ldots, \eta_n$ - isometric to it base in isometric space of random variables H^{+}. If projector on H^- is designated $\mathscr{P}$, then is the correlation function

$$B(t+s)=\mathsf{M}\xi(t)\overline{\xi(-s)}=(\xi(t),\ \xi(-s))=$$
$$=(\mathscr{P}\xi(t),\ \xi(-s))=\sum c_j(t)(\eta_j,\ \xi(-s)),\quad t,\ s\geqslant 0.$$

After designating $(\eta_j,\ \xi(-s))$ by $\mu_j(s)$, let us have

$$B(t+s)=\sum_{j=1}^{n} c_j(t)\mu_j(s). \qquad (4.11)$$

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Further, whatever numbers $0<t_0<\ldots<t_n<\infty$, random variables $\mathscr{P}\xi(t_0),\ \ldots,\ \mathscr{P}\xi(t_n)$ linearly dependent, so that will be located the numbers $a_0,\ a_1,\ \ldots,\ a_n$, for which

$$\sum a_j B(t_j+s)=(\sum a_j\mathscr{P}\xi(t_j),\ \xi(-s))\equiv 0. \qquad (4.12)$$

Equalities (4.11) and (4.12) already are sufficient in order to find the correlation function B (t). First, relying on (4.11), let us show that B (t) is infinitely differentiated with t > 0. Let us select n of infinitely differentiated functions $g_1(s)$, ..., $g_n(s)$ with carriers inside (0,∞) so, in order to $\det\left\|\int_0^\infty g_i(s)\mu_j(s)\,ds\right\|\neq 0.$ On the strength of (4.11)

$$\int_0^\infty B(t+s)g_i(s)\,ds=\sum_1^n c_j(t)\int_0^\infty g_i(s)\mu_j(s)\,ds.$$

The left sides of these equalities, equal to $\int_t^\infty B(u)g_i(u-t)\,du$, are infinitely differentiated on t. Therefore are infinitely differentiated everything $c_j(t)$, but on (4.11) and B (t), t > 0. By

differentiating identity (4.12), we will obtain

$$\sum_{l=0}^{n} a_l \frac{d^u}{ds^u} B(t_l + s) = 0, \quad u = 0, 1, 2, \ldots, n.$$

Consequently, it is possible to find this polynomial P (z) of degree not higher than n, that

$$P\left(\frac{d}{ds}\right) B(s) = 0, \quad s > 0. \qquad (4.13)$$

Well known (see [21], page 58), that the fundamental system of the solutions to equation (4.13) compose the functions of the form

$$e^{\lambda_1 s} R_1(s), \ \ldots, \ e^{\lambda_k s} R_k(s), \qquad (4.14)$$

where all numbers λ_l are different, a $R_l(s)$ - the polynomials of degree $n_j - 1$, whereupon $n_1 + \ldots + n_k$ are a degree of polynomial P ($\leqslant$ n). The correlation function B (s) - the solution to equation (4.13) - is a linear combination of functions (4.14).

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According to Rimana - Lebesgue's theorem with s → -

$$B(s) = \int_{-\infty}^{\infty} e^{i\lambda s} f(\lambda)\, d\lambda \to 0,$$

so that everything $\mathrm{Re}\,\lambda_l < 0$. Finally, B (s) = $\overline{B(-s)}$ for s < 0.

Direct calculations give

$$\int_0^\infty e^{-i\lambda s} e^{\lambda_l s} R_l(s)\,ds = \tilde{R}_l\left(\frac{1}{\lambda_l - i\lambda}\right), \qquad (4.15)$$

where $\tilde{R}_l$ is rational degrees not above 2n. Therefore

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda s} B(s)\,ds$$

- the rational integral function of degree is not above 2n.

In order to ascertain that degree f in accuracy is equal to 2n, sufficient to demonstrate the second part of the theorem: if f is the rational integral function of degree 2n, then dim $L^{+|-}$ (F) $\leqslant$ n.

This last/latter assertion is well known in the theory of forecast/prediction (see [22], of page 174). In order to demonstrate it, is decomposed rational integral function f (λ) into the sum of common fractions $1/(\lambda_l \pm i\lambda)^\alpha$, $\operatorname{Re}\lambda_l > 0$, but integer α does not exceed multiplicity n_l the conjugated/combined poles $\pm i\lambda_l$. By applying Fourier transform to (4.15), let us find that the correlation function B (s), s > 0, again is a sum of the functions of form (4.14) and, which means, of B (s) there is a solution to the linear differential equation of form (4.13) of degree not higher than n. Any n + 1 solutions B (t_0 + s), ..., $B(t_n + s)$, $t_l > 0$, equations (4.13) are linearly dependent, so that will be located the numbers a_l, for which

$$\sum_0^n a_i B(t_i + s) = 0, \qquad s > 0.$$

But then and

$$(\sum a_i \mathscr{P} \xi(t_i), \xi(-s)) = \sum_0^n a_i (\xi(t_i), \xi(-s)) = \sum_0^n a_i B(t_i + s) = 0$$

with all s > 0, therefore, $\sum_0^n a_i \mathscr{P} \xi(t_i) = 0$, i.e., any n + 1 vectors from $H^{+|-}$ (from $L^{+|-}$) are linearly dependent.

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The case of continuous time is dismantle/selected to end.

2. ξ (t) - process with discrete time. Proof similarly to that which was given is above and even somewhat simpler. Equality (4.12) occurs as before, but now here t_i and s are integers. Let us determine operator Δ the taking of difference by equality ΔB (s) = B (s + 1) - B (s). From (4.12) it follows that

$$\sum_0^n a_i \Delta^k B(t_i + s) = 0,$$

whence escape/ensues the analog of equation (4.13) - difference equation:

$$P(\Delta) B(s) = 0. \qquad (4.16)$$

From the theory of such equations [1] it is known that any solution to

equation (4.16) is the linear combination n of the linearly independent fundamental solutions, which have as before form (4.14), only this time in (4.14) s - whole.

FOOTNOTE [1]. See, for example, [7]. ENDFOOTNOTE.

Further reasonings completely analogous to those which were used with the analysis/selection of the continuous case, and we them lower. Theorem is demonstrated.

This theorem explains sufficiently well that role, which the rational spectral densities play the theory of forecast/prediction. Let f (λ) - is rational function λ (or $e^{i\lambda}$) the degrees 2n. It it is possible to register in the form $f=\left|\frac{P}{Q}\right|^2$, where $\frac{P}{Q}=g\in\mathcal{H}^{2+}$, Q and P - the polynomials of degree with respect to n and $n_1 \leqslant n-1$. It is not difficult to show that in the case of processes with continuous time the base of space $L^{+/-}$ form functions $\frac{\lambda^s}{P(\lambda)}$, $s=0,1,\ldots,n-1$, and in the case of processes with discrete time - function $\frac{e^{ij\lambda}}{P(e^{i\lambda})}$, $j=1,\ldots,n$. In particular (this follows, however, and from theorem 5), $L^{+/-}$ = L^0, only if P are const.

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Let us note that even if not to assume the spectral measure F absolutely continuously, we nevertheless will obtain that the correlation function B (t) is a sum of functions (4.14), but now possible that for some λ_l $\mathrm{Re}\,\lambda_l=0$ (of course, for these λ_l $R_l(s)\equiv\mathrm{const}$). The transformation of the Fourier function $e^{\lambda_l s}$, $\mathrm{Re}\,\lambda_l=0$, is δ-measure with load at point λ_l. After recalling p. 1 theorems 1, 2, we we will obtain the following common/general/total assertion:

$$\dim L^{+|-}(F)=n<\infty$$

in that and only that case, if $F=F_a+F_s$, where the derivative of absolutely continuous part F_a' there is rational function $\lambda(e^{i\lambda})$ degree $2n_1$, and singular measure F_s is concentrated in n_2 different points, $n_1+n_2=n$.

§5. Structure of σ-algebra of events $\mathfrak{A}$ (T).

In this paragraph we will show as some results relative to subspaces H (T) (or isometric with it subspaces $L_T(F)$) they give (in the case of Gaussian processes) to theorems relative to σ-algebra $\mathfrak{A}(T)$. Other results in this direction are contained in chapter IV.

Let us turn to relationship/ratios (4.1) and we will try ourselves to seek the analogs for these relationship/ratios and other results §4, expressed in the language of σ-algebra $\mathfrak{A}(T)$. It is understandable that to space $H^- = H(-\infty, 0)$ (or isometric to it to space $L^-(F)$) corresponds to σ-algebra of events $\mathfrak{A}^- = \mathfrak{A}(-\infty, 0)$, to space $H^+ = H(0, \infty)$ σ-algebra $\mathfrak{A}^+ = \mathfrak{A}(0, \infty)$, to space $H^0 = \bigcap_{t>0} H(-t, t) = \bigcap_{t>0} H(0, t)$ - σ-algebra $\mathfrak{A}^0 = \bigcap_{t>0} \mathfrak{A}(-t, t)$.

More complexly is matter with analog for space $H^{+|-}$ (or, that $L^{+|-}(F)$) - by projection in H of subspace H^+ on H^-. Preliminarily let us introduce the following concept. Determination [1]: splitting σ-algebra at point t for a process $\xi(t)$ is called any σ-algebra $\mathfrak{A}_1^{(t)} \subseteq \mathfrak{A}(-\infty, t)$, relative to which the past process $\mathfrak{A}(-\infty, t)$ and the future $\mathfrak{A}(t, \infty)$ are conditionally independent, i.e., for any $A \in \mathfrak{A}(-\infty, t), B \in \mathfrak{A}(t, \infty)$

$$P\{AB \mid \mathfrak{A}_1^{(t)}\} = P\{A \mid \mathfrak{A}_1^{(t)}\} P\{B \mid \mathfrak{A}_1^{(t)}\}.$$

FOOTNOTE [1]. Term "splitting σ-algebra" introduced Mckean (H P Mckean, Jr., Brownian motion with a several dimension time, is theoretically probable, and its use. 8 (1963), 357-378; see also the cited on page 62 article of Levinson and Mckean).

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It is not difficult to comprehend that splitting σ-algebra at point t always exists; for example, such will be σ-algebra $\mathfrak{A}(-\infty, t)$. It goes without saying that interest represents minimum splitting σ-algebra at point t (below we let us demonstrate that she always exists). For example, for Markov processes minimum splitting σ-algebra at point t is σ-algebra, generated by random variable ξ (t).

For stationary in narrow sense processes, in particular for stationary Gaussian processes, it suffices to examine only splitting σ-algebra at point by 0. Minimum splitting σ-algebra at point 0 we will designate $\mathfrak{A}^{+|-}$; as it follows from the following, precisely, this σ-algebra and there is a natural analog of space $H^{+|-}$.

Theorem 7. Let ξ (t) be a stationary Gaussian process with the spectral measure F (dλ). Always they occur of the connection/inclusion

$$\mathfrak{A}^{-} \supseteq \mathfrak{A}^{+|-} \supseteq \mathfrak{A}^{-} \cap \mathfrak{A}^{-} \supseteq \mathfrak{A}^{0}. \qquad (5.1)$$

For any of the signs $\supseteq$ in (5.1) can be replaced with equal sign in

that and only that case, if it is possible to replace with the equal sign of sign $\supseteq$, which stands on corresponding place in the relationships

$$H^{-} \supseteq H^{+|-} \supseteq H^{-} \cap H^{+} \supseteq H^{0} \qquad (5.2)$$

or, which is the same thing, in relationship/ratios (4.1):

$$L^{-}(F) \supseteq L^{+|-}(F) \supseteq L^{-}(F) \cap L^{+}(F) \supseteq L^{0}(F).$$

Proof let us begin from research of the common properties of splitting σ-algebra.

Lemma 3. Let ξ (t) be an arbitrary random process, $\mathfrak{A}^{-} = \mathfrak{A}(-\infty, 0)$, $\mathfrak{A}^{+} \doteq (0, \infty)$.

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Then:

1) $\mathfrak{A}^{-}$ there is splitting σ-algebra (at point 0);

2) if $\mathfrak{A}_1$ be splitting σ-algebra, and event $B \in \mathfrak{A}^{+}$, then $\mathsf{P}\{B \mid \mathfrak{A}^{-}\} = \mathsf{P}\{B \mid \mathfrak{A}_1\}$;

3) if $\mathfrak{A}_1$ is splitting σ-algebra and $\mathfrak{A}^{-} \supseteq \mathfrak{A}_2 \supseteq \mathfrak{A}_1$, then $\mathfrak{A}_2$ - also

splitting σ-algebra;

4) if σ-algebra $\mathfrak{A}_1, \mathfrak{A}_2$ - splitting, then splitting it is also σ-algebra $\mathfrak{A}_1 \cap \mathfrak{A}_2$;

5) there is minimum splitting σ-algebra $\mathfrak{A}^{+|-}$;

6) always $\mathfrak{A}^{+|-} \supseteq \mathfrak{A}^+ \cap \mathfrak{A}^-$.

Proof of lemma. Let us designate through χ_A the indicator of event A, i.e.,

$$\chi_A = \chi_A(\omega) = \begin{cases} 0, & \omega \notin A, \\ 1, & \omega \in A. \end{cases}$$

1) On the strength of the properties of conditional mathematical expectations for $A^- \in \mathfrak{A}^-$, $B \in \mathfrak{A}^+$ we have

$$\begin{aligned} \mathsf{P}\{AB \mid \mathfrak{A}^-\} &= \mathsf{M}\{\chi_A \cdot \chi_B \mid \mathfrak{A}^-\} = \chi_A \cdot \mathsf{M}\{\chi_B \mid \mathfrak{A}^-\} = \\ &= \mathsf{M}\{\chi_A \mid \mathfrak{A}^-\}\, \mathsf{M}\{\chi_B \mid \mathfrak{A}^-\} = \mathsf{P}\{A \mid \mathfrak{A}^-\}\, \mathsf{P}\{B \mid \mathfrak{A}^-\}. \end{aligned}$$

2) Regarding that which split σ-algebra $\mathfrak{A}_1 \subset \mathfrak{A}^-$, so that random variable $\mathsf{P}\{B \mid \mathfrak{A}_1\}$ measured relative to $\mathfrak{A}^-$. Therefore it suffices to demonstrate that the integrals of random variables $\mathsf{M}\{\chi_B \mid \mathfrak{A}_1\}$, $\mathsf{M}\{\chi_B \mid \mathfrak{A}^-\}$ in terms of any multitude $A \in \mathfrak{A}^-$ coincide, and to exile to Radon - Nicodemius' theorem. We have

$$\int_A \mathrm{M}\{\chi_B \mid \mathfrak{A}^-\}\, d\mathrm{P} = \mathrm{M}\{\chi_A \cdot \chi_B\} = \mathrm{M}\{\mathrm{M}\{\chi_A \cdot \chi_B \mid \mathfrak{A}_1\}\} =$$
$$= \mathrm{M}\{\mathrm{M}\{\chi_A \mid \mathfrak{A}_1\} \cdot \mathrm{M}\{\chi_B \mid \mathfrak{A}_1\}\} = \mathrm{M}\{\chi_A \cdot \mathrm{M}\{\chi_B \mid \mathfrak{A}_1\}\} =$$
$$= \int_A \mathrm{M}\{\chi_B \mid \mathfrak{A}_1\}\, d\mathrm{P}.$$

3) Let $A \in \mathfrak{A}^-$, $B \in \mathfrak{A}^+$. Since to both part of the predicted equality $\mathrm{M}\{\chi_A \cdot \chi_B \mid \mathfrak{A}_2\} = \mathrm{M}\{\chi_A \mid \mathfrak{A}_2\} \times \mathrm{M}\{\chi_B \mid \mathfrak{A}_2\}$ measured relative to $\mathfrak{A}_2$, is sufficient to again demonstrate that coincide the integrals of them in terms of any multitude $C \in \mathfrak{A}_2$.

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On the basis p. 2) $\mathrm{M}\{\chi_B \mid \mathfrak{A}_1\} = \mathrm{M}\{\chi_B \mid \mathfrak{A}^-\}$ and, therefore, $\mathrm{M}\{\chi_B \mid \mathfrak{A}_1\} = \mathrm{M}\{\chi_B \mid \mathfrak{A}_2\}$. Therefore

$$\int_C \mathrm{M}\{\chi_A\chi_B \mid \mathfrak{A}_2\}\, d\mathrm{P} = \mathrm{M}\{\chi_A\chi_B\chi_C\} =$$
$$= \mathrm{M}\{\mathrm{M}\{\chi_A\chi_C \mid \mathfrak{A}_1\} \cdot \mathrm{M}\{\chi_B \mid \mathfrak{A}_1\}\} =$$
$$= \mathrm{M}\{\mathrm{M}\{\mathrm{M}\{\chi_A\chi_C \mid \mathfrak{A}_2\} \cdot \mathrm{M}\{\chi_B \mid \mathfrak{A}_2\} \mid \mathfrak{A}_1\}\} =$$
$$= \mathrm{M}\{\chi_C\, \mathrm{M}\{\chi_A \mid \mathfrak{A}_2\} \cdot \mathrm{M}\{\chi_B \mid \mathfrak{A}_2\}\} =$$
$$= \int_C \mathrm{M}\{\chi_A \mid \mathfrak{A}_2\} \cdot \mathrm{M}\{\chi_B \mid \mathfrak{A}_2\}\, d\mathrm{P}.$$

4) Let us note first that for any random variable ζ $(\mathrm{M}\zeta = 0,\ \mathrm{M}|\zeta|^2 < \infty)$

occurs the equality

$$M\{\zeta \mid \mathfrak{A}_1 \cap \mathfrak{A}_2\} = M\{M\{\ldots M\{\zeta \mid \mathfrak{A}_1\} \mid \mathfrak{A}_2\} \mid \mathfrak{A}_1 \ldots\}. \quad (5.3)$$

Actually, let us consider hilbert space H all random variables with zero average, final dispersion and scalar product $(\eta_1, \eta_2) = M\eta_1\eta_2$. By H_i, $i = 1, 2$, let us designate the subspaces H, which consist of those values, which are measured relative to $\mathfrak{A}_i$. Then the operators $\mathscr{P}_i = M\{\cdot \mid \mathfrak{A}_i\}$ essence projectors in H on H_i. The left side of equality (5.3) is a projection ζ on $H_1 \cap H_2$, right side is the result of the application/use to ζ of an operator $\lim_n (\mathscr{P}_2\mathscr{P}_1)^n$, equal to projector on $H_1 \cap H_2$.

Let now $A \in \mathfrak{A}^-$, $B \in \mathfrak{A}^+$. Then

$$M\{\chi_A\chi_B \mid \mathfrak{A}_1\} = M\{\chi_A \mid \mathfrak{A}_1\} \cdot M\{\chi_B \mid \mathfrak{A}_1\}. \quad (5.4)$$

On the basis p. 2) and equality (5.3)

$$M\{\chi_B \mid \mathfrak{A}_1 \cap \mathfrak{A}_2\} = \lim_n \underbrace{M\{M\{M\{\ldots M}_{n\ \text{раз}}\{\chi_B \mid \mathfrak{A}_1\} \mid \mathfrak{A}_2\}\mathfrak{A}_1 \ldots\} =$$
$$= M\{\chi_B \mid \mathfrak{A}_1\} = M\{\chi_B \mid \mathfrak{A}_2\} = M\{\chi_B \mid \mathfrak{A}^-\}.$$

Therefore, if we use to both parts of equality (5.4) operator $M\{\cdot \mid \mathfrak{A}_1 \cap \mathfrak{A}_2\}$, we let us find

$$M\{\chi_A\chi_B \mid \mathfrak{A}_1 \cap \mathfrak{A}_2\} = M\{\chi_A M\{\chi_B \mid \mathfrak{A}_1\} \mid \mathfrak{A}_1 \cap \mathfrak{A}_2\} =$$
$$= M\{\chi_A \mid \mathfrak{A}_1 \cap \mathfrak{A}_2\} \cdot M\{\chi_B \mid \mathfrak{A}_1 \cap \mathfrak{A}_2\}. \quad (5.5)$$

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5) On the basis p. 1) many all splitting σ-algebra are not

empty. Let $\mathfrak{A}^{+|-}$ there is an intersection of all splitting σ-algebra. Let us demonstrate that $\mathfrak{A}^{+|-}$ there is split and, therefore, minimum splitting σ-algebra. On the strength of p. 2) all the probabilities $\mathsf{P}\{B\mid\mathfrak{A}_1\}$, $B\in\mathfrak{A}^+$, where $\wedge$ a $\mathfrak{A}_1$ - arbitrary splitting σ-algebra, they are indentical and, which means, are equal to $\mathsf{P}\{B\mid\mathfrak{A}^{+|-}\}$. After taking of both parts (5.4) conditional mathematical expectation under condition $\mathfrak{A}^{+|-}$, let us find analogously to (with 5.5)

$$\mathsf{M}\{\chi_A\chi_B\mid\mathfrak{A}^{+|-}\}=\mathsf{M}\{\chi_A\mid\mathfrak{A}^{+|-}\}\,\mathsf{M}\{\chi_B\mid\mathfrak{A}^{+|-}\},$$

i.e. $\mathfrak{A}^{+|-}$ is splitting σ-algebra.

6) Relationship/ratio $\mathfrak{A}^{+|-}\supseteq\mathfrak{A}^-\cap\mathfrak{A}^+$ is obvious. Lemma is demonstrated.

Lemma 4. Let ξ (t) be a stationary in the narrow sense of stochastic continuous process; then

$$\mathfrak{A}^{+|-}\supseteq\mathfrak{A}^+\cap\mathfrak{A}^-\supseteq\mathfrak{A}^0. \qquad (5.6)$$

Let us demonstrate first the left connection/inclusion. Let $B\in\mathfrak{A}^+\cap\mathfrak{A}^-$. Since $\mathfrak{A}^{+|-}$ - splitting σ-algebra,

$$\mathsf{P}\{B\mid\mathfrak{A}^{+|-}\}=\mathsf{P}\{BB\mid\mathfrak{A}^{+|-}\}=(\mathsf{P}\{B\mid\mathfrak{A}^{+|-}\})^2.$$

This equality means that the random variable $\mathsf{P}\{B\mid\mathfrak{A}^{+|-}\}$ takes the only two values: 0 and 1. Let $A=\{\omega\colon \mathsf{P}\{B\mid\mathfrak{A}^{+|-}\}=1\}$, and let $\bar{A}$ there is addition A. It is understandable that $A\in\mathfrak{A}^{+|-}$, $\bar{A}\in\mathfrak{A}^{+|-}$. Therefore

$$\int_A \mathsf{P}\{B \mid \mathfrak{A}^{+|-}\}\, d\mathsf{P} = \mathsf{P}\{AB\} = \mathsf{P}\{A\}.$$

$$\int_{\bar{A}} \mathsf{P}\{B \mid \mathfrak{A}^{+|-}\}\, d\mathsf{P} = \mathsf{P}\{\bar{A}B\} = 0.$$

On the basis of these equations B differs from A not more than by the event of probability zero, so that it is possible to count $B = A \in \mathfrak{A}^{+|-}$.

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Let us pass to the proof of right connection/inclusion (5.6). Let us assume $\mathfrak{A}^{0+} = \bigcap_{t>0} \mathfrak{A}(0, t)$, $\mathfrak{A}^{0-} = \bigcap_{t>0} \mathfrak{A}(-t, 0)$. It suffices to demonstrate that $\mathfrak{A}^{0+} = \mathfrak{A}^{0} = \mathfrak{A}^{0}$. Stochastic continuous process ξ (t) generates the group of the retaining the probability transforms $T^{t} = TA$, $A \in \mathfrak{A}(-\infty, \infty)$, which in turn, assigns the group of the continuous unitary operators [1] U^{t} on H.

FOOTNOTE [1]. See [22], page 206-211. ENDFOOTNOTE.

Let now A be any event from $\mathfrak{A}^{0}$. It is obvious, $T^{t}A \in \mathfrak{A}(0, t)$. Let, further, $\eta(t)$ there is an indicator of event $T^{t}A$. It is understandable that $\eta(t) = U^{t}\chi_{A}$ there is a stationary continuous on the average quadratic process. Therefore

$$\lim_{t\to 0} \mathsf{M}\,|\,\eta(t) - \eta(0)\,|^{2} = 0.$$

For all t > 0 random variables $\eta(t)$ are measurable relative to $\mathfrak{A}(0, t)$,

and therefore $\eta(0)$ is measurable relative to $\mathfrak{A}^{0+}$. Analogously it is proven, that $\eta(0)$ is measurable relative to $\mathfrak{A}^{0-}$. Consequently, $A \in \mathfrak{A}^{0+}$, $A \in \mathfrak{A}^{0-}$ and $\mathfrak{A}^{0} = \mathfrak{A}^{0+} = \mathfrak{A}^{0-}$.

Noting now that left connection/inclusion (5.1) is trivial, we see that together with lemma 4 is demonstrated the first part of the theorem. It remained to explain, when in (5.1) they can occur of equality. For this purpose we will demonstrate several lemmas relative to communication/connections between σ-algebra $\mathfrak{A}(T)$ and spaces H (T). Let us agree by means $A(\Xi)$ to designate minimum σ-algebra of events, generated by many random variables Ξ, thus, for instance, $\mathfrak{A}(T) = A(\xi(t), t \in T)$. It is obvious, further that $\mathfrak{A}^{-} = A(H^{-})$.

Lemma 5. For a stationary Gaussian process minimum splitting σ-algebra is

$$\mathfrak{A}^{+|-} = A(H^{+|-}).$$

Proof. Let us demonstrate first that $\mathfrak{A}^{+|-} \supseteq A(H^{+|-})$. Let us designate by $\mathcal{P}^{-} = \mathcal{P}$ the projector in H (-∞, ∞) on H. It is clear that $\mathcal{P} = \mathsf{M}\{\cdot \mid \mathfrak{A}^{-}\}$.

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Random variable ξ (t), t ≥ 0, can be approximated as conveniently

good on the average quadratic by the linear combinations of indicators χ_B, $B \in \mathfrak{A}^+$. Therefore on the basis p. 2) lemma 3

$$\mathscr{P}\xi(t) = \mathsf{M}\{\xi(t) \mid \mathfrak{A}^-\} = \mathsf{M}\{\xi(t) \mid \mathfrak{A}^{+|-}\}.$$

Consequently, all cell/elements of space $H^{+|-}$ are measurable relative to $\mathfrak{A}^{+|-}$ and $A(H^{+|-}) \subseteq \mathfrak{A}^{+|-}$.

Let us show now that $A(H^{+|-})$ splits σ-algebra $\mathfrak{A}^+$, $\mathfrak{A}^-$. Since $\mathfrak{A}^{+|-}$ - minimum splitting σ-algebra, hence and from previous will follow that $\mathfrak{A}^{+|-} = A(H^{+|-})$. Let χ_A - be an indicator of event $A \in \mathfrak{A}^-$; let the random variable η, measurable relatively $\mathfrak{A}^+$, representable in the form of product $\eta = \eta_1\eta_2$, where random variable η_1 is measurable relatively $A(H^{+|-})$, and random variable η_2 is independent of σ-algebra $\mathfrak{A}^-$. Let us demonstrate that

$$\mathsf{M}\{\chi_A\eta \mid A(H^{+|-})\} = \mathsf{M}\{\chi_A \mid A(H^{+|-})\} \cdot \mathsf{M}\{\eta \mid A(H^{+|-})\}. \quad (5.7)$$

Actually,

$$\begin{aligned}\mathsf{M}\{\chi_A\eta \mid A(H^{+|-})\} &= \eta_1\mathsf{M}\{\chi_A\eta_2 \mid A(H^{+|-})\} = \\ &= \mathsf{M}\{\eta_1 \mid A(H^{+|-})\} \cdot \mathsf{M}\{\chi_A \mathsf{M}\{\eta_2 \mid \mathfrak{A}^-\} \mid A(H^{+|-})\} = \\ &= \mathsf{M}\{\eta_1 \mid A(H^{+|-})\} \cdot \mathsf{M}\eta_2 \cdot \mathsf{M}\{\chi_A \mid A(H^{+|-})\} = \\ &= \mathsf{M}\{\eta_1\eta_2 \mid A(H^{+|-})\} \cdot \mathsf{M}\{\chi_A \mid A(H^{+|-})\}.\end{aligned}$$

Arbitrary random variable $\xi(t)$, $t \geqslant 0$, can be registered in the

form of sum ξ (t) = η_1 (t) + η_2 (t), where random variable $\eta_1(t) = \mathscr{P}\xi(t) = \mathrm{M}\{\xi(t) \mid \mathfrak{A}^-\} \in H^{+|-}$ and, which means, is measurable relatively $A(H^{+|-})$, but random variable η_2 (t) is orthogonal H^-. Orthogonal Gaussian values are independent, so that η_2 (t) does not depend on all values from H^-, and also, therefore, on σ-algebra $\mathfrak{A}^-$.

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Let now $Q = Q(\xi(t_1), \ldots, \xi(t_n))$ -be an arbitrary polynomial of values $\xi(t_1), \ldots, \xi(t_n)$, $t_i \geqslant 0$. Record/writing each value $\xi(t_i)$ in the form of sum $\eta_1(t_i) + \eta_2(t_i)$, we let us will be able polynomial itself Q to rewrite as bag of form $\sum_l \zeta_l$, where each $\zeta_l = \eta_{1l} \cdot \eta_{2l}$, η_{1l} is measurable relative to $A(H^{+|-})$, and η_{2l} does not depend on $\mathfrak{A}^-$. On the strength of (5.7) for any such polynomial Q

$$\mathrm{M}\{\chi_A \cdot Q \mid A(H^{+|-})\} = \\ = \mathrm{M}\{\chi_A \mid A(H^{+|-})\} \cdot \mathrm{M}\{Q \mid A(H^{+|-})\}. \quad (5.8)$$

Let now B be an event from $\mathfrak{A}^+$. As proved in §5 chapter I, any measurable relatively $\mathfrak{A}^+$ random variable with the final dispersion, in particular random variable χ_B, is a limit on the average quadratic polynomials Q above form indicated. On the basis (5.8)

$$\mathrm{M}\{\chi_A \cdot \chi_B \mid A(H^{+|-})\} = \mathrm{M}\{\chi_A \mid A(H^{+|-})\} \cdot \mathrm{M}\{\chi_B \mid A(H^{+|-})\},$$

i.e. $A(H^{+|-})$- splitting σ-algebra. Lemma is demonstrated.

Lemma 6. For the stationary Gaussian process

$$\mathfrak{A}^+ \cap \mathfrak{A}^- = A(H^+ \cap H^-).$$

It suffices to demonstrate that for any polynomial P of the random variables $\xi(t_1), \ldots, \xi(t_n)$

$$M\{P \mid \mathfrak{A}^+ \cap \mathfrak{A}^-\} = M\{P \mid A(H^+ \cap H^-)\}. \qquad (5.9)$$

Let us prove this equality by induction. For n = 1 P = ξ (t) there is a cell/element of space H (-∞, ∞). Further, it is analogous to (with 5.3), for all random variables $M\zeta^2 < \infty$, occurs the equality

$$M\{\zeta \mid \mathfrak{A}^+ \cap \mathfrak{A}^-\} = \lim_n \underbrace{M\{M\{\ldots M}_{2n \text{ times}}\{\zeta \mid \mathfrak{A}^+\} \mid \mathfrak{A}^-\} \ldots \mid \mathfrak{A}^-\}. \quad (5.10)$$

It is obvious, $\mathfrak{A}^+ = A(H^+)$, $\mathfrak{A}^- = A(H^-)$, so that if h be cell/element H (-∞, ∞), then [1] $M\{h \mid \mathfrak{A}^+\} \in H^+ \subset H(-\infty, \infty)$.

FOOTNOTE [1]. See §5 chapter I. ENDFOOTNOTE.

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Hence and from (5.10) it follows that also $M\{h \mid \mathfrak{A}^- \cap \mathfrak{A}^+\} \in H(-\infty, \infty)$ for all $h \in H(-\infty, \infty)$. Since random variable $M\{h \mid \mathfrak{A}^- \cap \mathfrak{A}^+\}$ is measurable relative to $\mathfrak{A}^-$, it belongs H^-; in exactly the same manner $M\{h \mid \mathfrak{A}^- \cap \mathfrak{A}^+\} \in H^+$. Thus,

equality (5.9) is proved for the polynomials of the first degree.

The remaining part of the proof is similar to reasonings §5 chapter I. Let us assume that equality (5.9) is proved for all polynomials P of degree not higher than n − 1, n ⩾ 2, and let us demonstrate it for the polynomials of degree n. It suffices to consider polynomials P of form $\xi(t_1)\ldots\xi(t_n)$. Let us agree to designate through $\mathcal{P}^+$, $\mathcal{P}^-$ operators $M\{\cdot|\mathfrak{A}^+\}$, $M\{\cdot|\mathfrak{A}^-\}$ (these operators they are projectors in H on H^+, H^- respectively). If $\xi(t_i)=\mathcal{P}^+\xi(t_i)+\eta(t_i)=\xi_1(t_i)+\eta(t_i)$, that all random values $\eta(t_i)$ are orthogonal H^+ and, which means, are independent of $\mathfrak{A}^+$. We have (see §5 chapter I)

$$M\{P|\mathfrak{A}^+\}=\xi_1(t_1)\ldots\xi_1(t_n)+Q_1,$$

where Q_1 there is a limit on the average quadratic the polynomials of degree not higher than n - 1. By set/assuming, further, $\xi_1(t_i)=\mathcal{P}^-\xi_1(t_i)+\eta_1(t_i)$ $=\xi_2(t_i)+\eta_1(t_i)$, we will obtain that

$$M\{\xi_1(t_1)\ldots\xi_1(t_n)+Q_1|\mathfrak{A}^-\}=\xi_2(t_1)\ldots\xi_2(t_n)+Q_2,$$

where again Q_2 there is a limit on the average quadratic the polynomials of degree not higher than n - 1. By continuing analogously, let us find finally that

$$\begin{aligned}M\{P|\mathfrak{A}^+\cap\mathfrak{A}^-\}&=\\ =\prod_{i=1}^{n}(\ldots\mathcal{P}^-\mathcal{P}^+\xi(t_i))+Q&=\prod_{i=1}^{n}M\{\xi(t_i)|\mathfrak{A}^+\cap\mathfrak{A}^-\}+Q=\\ &=\prod_{i}^{n}M\{\xi(t_i)|A(H^+\cap H^-)\}+Q.\end{aligned}$$

Here the first component to the right, obviously, is measurable relative to $A(H^+ \cap H^-)$, the second is is measurable relative to the same σ-algebra on induction assumption. Equality (5.9), and with it and lemma are demonstrated.

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Lemma 7. For the stationary Gaussian process

$$\mathfrak{A}^0 = A(H^0). \qquad (5.11)$$

Proof. It is obvious, $\mathfrak{A}^0 \supseteq A(H^0)$. Reversal is proven just as in lemma 6. Specifically, by induction according to the degree of polynomials P of $\xi(t_1) \cdots \xi(t_n)$ it is proven, that for any polynomial

$$\mathrm{M}\{P \mid \mathfrak{A}^0\} = \mathrm{M}\{P \mid A(H^0)\}. \qquad (5.12)$$

From the last/latter equality already follows the validity of equality (5.11). Induction passage of n - 1 to n is realized by those method, as in lemma 6. Therefore we only will demonstrate equality

(5.12) for the polynomials of the first degree. Let $P=h\in H(-\infty,\infty)$. Then all random variables $M\{h|\mathfrak{A}(0,1/n)\}$ and their limits on the average quadratic $M\{h|\mathfrak{A}^0\}$ essence the cell/elements of space H (-∞, ∞). Further, since all values $M\{h|\mathfrak{A}(0,s)\}, s<t,$ belong to space $H(0,s)\subset H(0,t),$

that and $M\{h|\mathfrak{A}^0\}$ belongs by all H (0, t), i.e., $M\{h|\mathfrak{A}^0\}\in H^0$ and, which means, $M\{h|\mathfrak{A}^0\}=M\{h|A(H^0)\}$. Lemma is demonstrated.

With the help of lemmas 5-7 second part of the theorem, which concerns the conditions of equality in relationship/ratios (5.1), is proven immediately. Let us demonstrate for a definition that $\mathfrak{A}^{+|-}=\mathfrak{A}^0$ in that and only that case, if $H^{+|-}=H^0$; other cases are examined analogously. Let $H^{+|-}=H^0$, then on the basis of lemmas $\mathfrak{A}^{+|-}=A(H^{+|-})=$ $=A(H^0)=\mathfrak{A}^0.$) Conversely, let $H^{+|-}\neq H^0$. Then $H^{+|-}\supset H^0$ there is a random variable $h\in H^{+|-}$,) which is orthogonal to space H^0. On the strength of Gaussian nature of random variable h is independent of all values from H, and also, therefore, of σ-algebra $A(H^0)=\mathfrak{A}^0$. Therefore event $\{h<0\}\in A(H^{+|-})=\mathfrak{A}^{+|-}$

does not depend on σ-algebra $\mathfrak{A}^0$ and cannot belong to it. Theorem is demonstrated.

The results of the previous paragraph together with recently the demonstrated theorem make it possible to express in the spectral terms of the conditions, under which in (5.1) instead of the connection/inclusions they occur of equality.

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For example, let ξ (t) - stationary Gaussian process with spectral density f (λ), then σ-algebra $\mathfrak{N}^0$ coincides with minimum splitting σ-algebra $\mathfrak{N}^{+|-}$ in that and only that case, if 1/f (λ) there is the integral function of zero degree. Appropriate to note also that the random process ξ (t) is Markov, if with all t minimum splitting σ-algebra at point t coincides with the algebra, generated by random variable ξ (t); therefore the process ξ (t) is Markov then and only then, if 1/f (λ) there is a polynomial of degree 2 (and m-component Markov, if 1/f (λ) there is a polynomial of degree 2m).

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Chapter III.

EQUIVALENT GAUSSIAN DISTRIBUTIONS AND THEIR DENSITIES.

§1. Some preliminary observations.

1. Introduction. Let $\xi = \xi(t)$ be a Gaussian random function of the parameter $t \in T$ with values $\xi(t) = \xi(\omega, t)$, $\omega \in \Omega$, on arbitrary probability space $(\Omega, \mathfrak{A}, P)$. Let us consider that σ-algebra $\mathfrak{A}$ is generated by all values $\xi(t) = \xi(\omega, t)$ on Ω (parameter t passes set T), so that the probability measure P in σ-algebra $\mathfrak{A} = \mathfrak{A}_\xi$ is Gaussian.

Let P_1 - be another Gaussian measure in σ-algebra $\mathfrak{A}$ [1].

FOOTNOTE [1]. It is more precise, measure P_1 is such, that the random function $\xi(t)$ on $(\Omega, \mathfrak{A}, P_1)$ is Gaussian. ENDFOOTNOTE.

It is called absolutely continuous **relatively** P. **if** $P_1(A)=0$ **with** $P(A)=0$, $A\in\mathfrak{A}$. As is known, absolutely **continuous measure** P_1 **is** representable in the form

$$P_1(A)=\int_A p(\omega)P(d\omega), \qquad A\in\mathfrak{A}, \qquad (1.1)$$

where p (ω) the determined nonnegative function on Ω, called density and designated $p(\omega)=P_1(d\omega)/P(d\omega)$. Measures P_1 and P are called equivalent, if they are mutually absolutely continuous. Measures P_1 and P are called orthogonal, if there are the nonintersecting multitudes A and $A_1\in\mathfrak{A}$ (called the carriers of the corresponding measures P and P_1), for which

$$P(A)=1, \quad P(A_1)=0$$

and

$$P_1(A)=0, \quad P_1(A_1)=1. \qquad (1.2)$$

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Absolute continuity means that, whatever $\varepsilon>0$, will be located such δ > with 0 that

$$P_1(A)\leqslant\varepsilon \text{ with } P(A)\leqslant\delta \qquad (1.3)$$

for all $A\in\mathfrak{A}$. This, for example, immediately follows from the inequality

$$P_1(A)=\int_A p(\omega)P(d\omega)\leqslant NP(A)+\int_{|p(\omega)|>N} p(\omega)P(d\omega),$$

where

$$\int_{|p(\omega)|>N} p(\omega)\,\mathsf{P}(d\omega) \leqslant \frac{\varepsilon}{2}$$

with sufficiently large N, so that, for example, $P_1(A) \leqslant \varepsilon$ with $\mathsf{P}(A) \leqslant \varepsilon/(2N)$.

Any measure P_1 representable [1] in the form of the sum of the orthogonal measures P_1' and P_1'', from which P_1' is orthogonal in measure P, and P_1'' is absolutely continuous relatively P.

FOOTNOTE [1]. See, for example, [8], page 111. ENDFOOTNOTE.

Accordingly

$$\mathsf{P}_1(A) = \mathsf{P}_1'(A) + \int_A p''(\omega)\,\mathsf{P}(d\omega), \qquad (1.4)$$

where

$$p''(\omega) = \mathsf{P}_1''(d\omega)/\mathsf{P}(d\omega).$$

Let us note that the measures P and P_1 are orthogonal, if for certain sequence of sets $A_n \in \mathfrak{A}$, $n = 1, 2, \ldots$, are fulfilled the relationship/ratios

$$\lim_{n\to\infty} \mathsf{P}(A_n) = 0, \quad \lim_{n\to\infty} \mathsf{P}_1(A_n) = 1. \qquad (1.5)$$

This, for example, follows from the fact that for nonorthogonal measures $P_1''(\Omega)>0$ and $P_1'(\Omega)<1$, and therefore under condition $\lim_{n\to\infty} P(A_n)=0$ we have

$$\lim_{n\to\infty} P_1(A_n)=\lim_{n\to\infty} P_1'(A_n)+\lim_{n\to\infty} P_1''(A_n)=$$
$$=\lim_{n\to\infty} P_1'(A_n)\leqslant P_1'(\Omega)<1.$$

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In the different regions of the theory of random processes and mathematical statistics, information theory, etc. do arise the questions: when the assigned measures P_1 and P are equivalent (or orthogonal)? as to compute density $p(\omega)=P_1(d\omega)/P(d\omega)$ equivalent measures? as to clearly describe the nonintersecting "carriers" of A and A_1 orthogonal measures?

Obviously (see §2 chapter I), during the solution to the placed above questions it is possible of Gaussian measures $P(d\omega)$ and $P_1(d\omega)$ in the initial space Ω to pass to the appropriate Gaussian distributions $P(dx)$ and $P_1(dx)$ in the containing all trajectories $\xi(\omega,\ \cdot)=\xi(\omega,\ t)$

1 function space $(X,\ \mathfrak{B})$ the real functions x = x (t) of the parameter $t\in T$, where σ-algebra $\mathfrak{B}$ is generated by all cylindrical multitudes of this space (see formulas (2.1) - (2.3) chapter I).

Specifically, always it is possible to pass to probability distributions P and P_1 in space $X=R^T$ all real functions x = x (t) of $t\in T$.

For any values $\varphi=\varphi(x)$ on X, measurable relative to σ-algebra $\mathfrak{B}$, value $\eta=\varphi[\xi(\omega,\ \cdot)]$ on Ω will be measurable relative to σ-algebra $\mathfrak{A}$; in this case for the integrated value $\varphi=\varphi(x)$ that which is integrated will be value η, whereupon

$$\int\limits_X \varphi(x)\,\mathsf{P}(dx)=\int\limits_\Omega \eta[\xi(\omega,\ \cdot)]\,\mathsf{P}(d\omega)=\mathsf{M}\eta. \qquad (1.6)$$

Moreover, in the case of a measurable multitude ξ (Ω) any measurable value η representable in the form $\eta=\varphi[\xi(\omega,\ \cdot)]$, where $\varphi=\varphi(x)$ is certain measurable value on $(X,\ \mathfrak{B})$.

Actually, as noted in §2 chapter I, any multitude A is prototype with representation $\xi=\xi(\omega,\ \cdot)$ from Ω in X: $A=\{\xi\in B\}$, where B, certain multitude from σ-algebra $\mathfrak{B}$. Therefore for any measurable multitudes $A_k=\{\xi\in B_k\}$ value $\eta=\sum_k c_k\chi_{A_k}(\omega)$ (where χ_A – an indicator of set A, i.e., $\chi_A(\omega)=1$ with $\omega\in A$ and $\chi_A(\omega)=0$ with $\omega\notin A$) representable in the form $\eta=\sum_k c_k\chi_{B_k}[\xi(\omega,\ \cdot)]=\varphi[\xi(\omega,\ \cdot)]$.

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Any measurable value $\eta=\eta(\omega)$ is (uniform on $\omega\in\Omega$) the limit of values

$\eta_n = \varphi_n[(\xi(\omega, \cdot)]$ the form indicated. It is obvious, $\eta = \varphi[\xi(\omega, \cdot)]$, where the function $\varphi = \varphi(x)$ on a measurable multitude ξ (Ω) is defined as (uniform on $x \in \xi(\Omega)$) the limit appropriate functions $\varphi_n = \varphi_n(x)$.

From equality (1.6) it escape/ensues, in particular, that if $p(x) = P_1(dx)/P(dx)$ – density of distribution P_1 and P, that $p[\xi(\omega, \cdot)] = P_1(d\omega)/P(d\omega)$ – the density of the initial probability measures on space Ω, since any multitude $A \in \mathfrak{A}$ representable in the form $A = \{\xi \in B\}$, where $B \in \mathfrak{B}$, and according to (1.6)

$$P_1(B) = \int_B p(x)P(dx) = \int_A p[\xi(\omega, \cdot)]P(d\omega) = P_1(A).$$

Let us note that the equivalent random functions ξ (t) and $\tilde{\xi}$ (t) have one and the same probability distribution in the appropriate function space $X = R^T$, so that, for example, if

$$\xi(\omega, t) = \tilde{\xi}(\omega, t) \qquad (1.7)$$

for almost all $\omega \in \Omega$ (relatively $P(d\omega)$ and $P_1(d\omega)$), then probability measures $P(d\omega)$ and $P_1(d\omega)$ in σ-algebra $\mathfrak{A}_\xi$ are equivalent or orthogonal when and only when this same property they possess in σ-algebra $\mathfrak{A}_{\tilde{\xi}}$ (generated by values $\tilde{\xi}(t)$, $t \in T$).

In connection with condition (1.7) it is worth saying also that addition/completion ¹ σ-algebra $\mathfrak{A}_\xi$ and $\mathfrak{A}_{\tilde{\xi}}$ by the sets of measure 0 (relative to equivalent P and P_1) it is one and the same σ-algebra $\mathfrak{A}^*$, whereupon the density $p(\omega) = P_1(d\omega)/P(d\omega)$ on $\mathfrak{A}_\xi$ (or $\mathfrak{A}_{\tilde{\xi}}$) simultaneously

is density, also, in full/total/complete σ-algebra $\mathfrak{A}^*$: for any multitude $A^* \in \mathfrak{A}^*$

$$\int_{A^*} p(\omega)\,\mathsf{P}(d\omega) = \int_{A} p(\omega)\,\mathsf{P}(d\omega) = \mathsf{P}_1(A) = \mathsf{P}_1(A^*), \qquad (1.8)$$

where the set $A \in \mathfrak{A}_\xi$ (or $A \in \mathfrak{A}_\xi$) is selected so, in order to for the symmetrical difference $A^* \circ A = (A^* \setminus A) \cup (A \setminus A^*)$

$$\mathsf{P}(A^* \circ A) = \mathsf{P}_1(A^* \circ A) = 0.$$

FOOTNOTE [1]. See, for example, [8], page 127. ENDFOOTNOTE.

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As has already been indicated in §2 chapter I, any Gaussian measure P is assigned by its average value of a (t), $t \in T$, and by correlation function $B(s, t)$, $s, t \in T$. The placed above problems (connected with the equivalency of Gaussian measures P and P_1) is logical to solve, proceeding precisely from the assigned average values of a (t), a_1 (t) and correlation functions B (s, t), B_1 (s, t) examine/considered Gaussian distribution P and P_1.

Obviously, without limiting generality, it is possible to count

that a (t) ≡ 0, since of values ξ (t) = ξ (ω, t) always it is possible to pass to values $\xi(t)-a(t), t \in T$, since during this passage σ-algebra $\mathfrak{A}$ and initial Gaussian measures $P(d\omega)$ and $P_1(d\omega)$ remain without change. In the appropriate function space this X indicates the passage to Gaussian distributions $P(dx)$ and $P_1(dx)$ with the previous correlation functions, that has average values of 0 and $a_1(t)-a(t),\ t \in T$.

2. Examples of orthogonal distributions. Let (relative to probability measure P) $\xi(t),\ 0 \leqslant t \leqslant \tau$, - Gaussian stationary process with the zero average and correlation function B (t). Let P_1 - be another probability measure, relative to which ξ (t) - also Gaussian stationary process with the zero average, but correlation function B_1 (t). Both these measures are examined lower in σ-algebra $\mathfrak{A}$, generated by all values ξ (t) = ξ (ω, t) at $t \in T$, where T = [0, τ] are certain segment on real straight line.

The simplest examples of orthogonal probability measures P and P_1 can be obtained, by taking into account the local properties of trajectories (see §7 chapter I).

For example, if

$$\varlimsup_{h \to 0} \frac{\Delta_{-h}\Delta_h B(0)}{\Delta_{-h}\Delta_h B_1(0)} = \infty$$

or

$$\lim_{h\to 0}\frac{\Delta_{-h}\Delta_h B(0)}{\Delta_{-h}\Delta_h B_1(0)} = 0, \tag{1.9}$$

that probability measures P and P_1 are orthogonal.

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In fact, if it is carried out, let us say, the second of conditions (1.9), then for certain function δ (h) such, that with h → 0

$$\lim_{h\to 0}\frac{\Delta_{-h}\Delta_h B(0)}{\delta(h)} = 0 \quad \textbf{and} \quad \lim_{h\to 0}\frac{\Delta_{-h}\Delta_h B_1(0)}{\delta(h)} = \infty,$$

with fixed/recorded t with probability 1 occur the following relationship/ratios:

$$\lim_{h\to 0}\frac{|\Delta_h\xi(t)|}{\delta^{1/2}(h)} = \begin{cases} 0 & \textbf{relative to } P, \\ \infty & \quad\text{''} \qquad \text{''} \quad P_1. \end{cases}$$

for fast enough descending sequence $h = h_n,\ n = 1, 2, \ldots$ (recall here, that $\Delta_{-h}\Delta_h B(0) = \mathsf{M}[\Delta_h\xi(t)]^2$). It is evident that the probability measures P and P_1 have the nonintersecting carriers A and A_1 form

$$A = \left\{\omega:\ \lim_{h=h_n\to 0}\frac{|\Delta_h\xi(\omega, t)|}{\delta^{1/2}(h)} = 0\right\}$$

and

$$A_1 = \left\{ \omega: \lim_{h-h_n \to 0} \frac{|\Delta_h \xi(\omega, t)|}{\delta^{1/2}(h)} = \infty \right\}:$$

Thus,

with the disturbance of condition [1]

$$\frac{\Delta_{-h}\Delta_h B(0)}{\Delta_{-h}\Delta_h B_1(0)} \asymp 1 \qquad (1.10)$$

the probability measures P and P_1 are orthogonal.

FOOTNOTE [1]. Relationship/ratio $\alpha \asymp \beta$ for variables α and β means that $0 < c_1 \leqslant \alpha/\beta \leqslant c_2 < \infty$ with some constant c_1 and c_2. ENDFOOTNOTE.

By taking into account the properties of trajectories, described in theorem 5 chapters I, it is possible to obtain very common/general/total examples of orthogonal distributions P and P_1.

Let us assume

$$b(t) = B(t) - B_1(t).$$

For simplicity of formulas let us consider that is fulfilled relationship/ratio (1.10). Let us show that

with the disturbance of the condition

$$\Delta_{-h}\Delta_h b(0) = o(\Delta_{-h}\Delta_h B(0)) \qquad (1.11)$$

the Gaussian measures P and P_1 are orthogonal.

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In fact, according to theorem 5 chapters I value

$$\eta(h)=\left[1-\frac{1}{N}\sum_{k=0}^{N-1}\frac{[\Delta_h\xi(kh)]^2}{\Delta_{-h}\Delta_h B(0)}\right]\frac{\Delta_{-h}\Delta_h B(0)}{\Delta_{-h}\Delta_h b(0)}=$$

$$=\left[1-\frac{1}{N}\sum_{k=0}^{N-1}\frac{[\Delta_h\xi(kh)]^2}{\Delta_{-h}\Delta_h B_1(0)}\right]\frac{\Delta_{-h}\Delta_h B_1(0)}{\Delta_{-h}\Delta_h b(0)}+1$$

$$(\text{где } N=[\tau/h])$$

are such, which with fast enough descending sequence $h=h_n$, $n=1, 2, \ldots,$ for which

$$\frac{\Delta_{-h}\Delta_h B(0)}{\Delta_{-h}\Delta_h b(0)}\asymp\frac{\Delta_{-h}\Delta_h B_1(0)}{\Delta_{-h}\Delta_h b(0)}\asymp 1,$$

with probability 1

$$\lim_{h\to 0}\eta(h)=\begin{cases}0 & \textbf{relative to } P,\\ 1 & \quad ''\qquad\quad '' \quad P_1.\end{cases}\qquad (1.12)$$

It is evident that with the disturbance of condition (1.11) the

Gaussian measures P and P_1 will be orthogonal.

Let us show still that

Gaussian measures P and P_1 orthogonal with the disturbance of the condition

$$\Delta_{-h}\Delta_h b(0) = O\left\{\frac{1}{N^2}\sum_{k,\,j=0}^{N-1}[\Delta_{-h}\Delta_h B((k-j)h)]^2\right\}^{1/2}. \quad (1.13)$$

In fact, above values $\eta(h)$ have relative to distributions P and P_1 average values 0 and 1, but dispersion (see formula (7.14) chapter I) the essence

$$\frac{\frac{2}{N^2}\sum_{k,\,j=0}^{N-1}[\Delta_{-h}\Delta_h B((k-j)h)]^2}{[\Delta_{-h}\Delta_h b(0)]^2} \text{ and } \frac{\frac{2}{N^2}\sum_{k,\,j=0}^{N-1}[\Delta_{-h}\Delta_h B_1((k-j)h)]^2}{[\Delta_{-h}\Delta_h b(0)]^2},$$

so that with the disturbance of condition (1.13) will be located sequence $h=h_n$, $n=1, 2, \ldots,$ for which will be carried out asymptotic relations (1.12).

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In this case Gaussian measures P and P_1 have the nonintersecting carriers of the form

$$A=\left\{\omega: \lim_{h_n\to 0}\eta(\omega,\ h_n)=0\right\}$$

and

$$A_1=\left\{\omega: \lim_{h_n\to 0}\eta(\omega,\ h_n)=1\right\}.$$

Recall that in lemma 4 chapters I is given estimation of expression $\sum_{k,\,j=0}^{N-1}[\Delta_{-h}\Delta_h B((k-j)h)]^2$. According to this estimation and the obtained above results

the Gaussian measures P and P_1 will be orthogonal with the disturbance of the following conditions:

$$\Delta_{-h}\Delta_h b(0) = O\{|h|^{1/2}|\Delta_{-h}\Delta_h B(0)|\} \qquad (1.14)$$

in the case, when

$$\frac{h}{\Delta_{-h}\Delta_h B(0)} = O(1), \qquad (1.15)$$

and

$$\Delta_{-h}\Delta_h b(0) = O\{h[\Delta_{-h}\Delta_h B(0)]^{1/2}\} \qquad (1.16)$$

otherwise.

From relationship/ratios (1.14), (1.16) it is possible to deduce the spectral conditions, under which Gaussian measures P and P_1 will be orthogonal. Specifically, let stationary process ξ (t) has spectral density f (λ) relatively P and f_1 (λ) relatively P_1. Then (see Section 1 §7 chapter I) at the condition

$$\lim_{\lambda\to\infty} f(\lambda)|\lambda|^{\alpha} = 0 \qquad (1.17)$$

value $\Delta_{-h}\Delta_h B(0)$ is such, that

$$\lim_{h\to 0}\frac{\Delta_{-h}\Delta_h B(0)}{h^{\alpha-1}} = 0,$$

and at the condition

$$\lim_{\lambda\to\infty}|f(\lambda)-f_1(\lambda)||\lambda|^{\beta} = \infty \qquad (1.18)$$

(where f (λ) $\geqslant$ f_1 (λ)) value $\Delta_{-h}\Delta_h b(0)$ satisfies the relationship/ratio

$$\lim_{h \to 0} \frac{\Delta_{-h}\Delta_h b(0)}{|h|^{\beta-1}} = \infty.$$

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It is evident that with $\beta \leqslant \alpha + \frac{1}{2}$ is broken the condition

$$\Delta_{-h}\Delta_h b(0) = O\{|h|^{1/2}|h|^{\alpha-1}\},$$

a by the facts more condition (1.14). Consequently,

for spectral densities f (λ) and f_1 (λ), that satisfy relationship/ratios (1.17) and (1.18), where $1 < \alpha \leqslant 2$ and $\beta \leqslant \alpha + \frac{1}{2}$, the Gaussian measures P and P_1 are orthogonal.

In conclusion let us consider one additional example of the orthogonal measures P and P_1, the correlation daies lily B (t) and B_1 (t) which they satisfy the conditions of theorem 6 chapters I, whereupon at certain point t of interval T = (0, τ) the derivatives B' (t) and B'_1 (t) have the "jumps" of the different value:

$$B'(t-0) - B'(t+0) \neq B'_1(t-0) - B'(t-0)$$

(with such correlation functions is given on page 144). According to theorem 6 chapters I in this case the nonintersecting carriers of Gaussian measures P and P_1 are the sets

$$A=\left\{\omega:\lim_{h_n\to 0}\eta(\omega,\ h_n)=B'(t-0)-B'(t+0)\right\}$$

and

$$A_1=\left\{\omega:\lim_{h_n\to 0}\eta(\omega,\ h_n)=B_1'(t-0)-B_1'(t+0)\right\},$$

where

$$\eta(h)=h^{-1}\frac{1}{N}\sum_{k=0}^{N-1}\Delta_h\xi(kh)\,\Delta_h\xi(t+kh)$$

and h_n, $n=1, 2, \ldots,$ - fast enough descending sequence.

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3. Some initial information about equivalent Gaussian distributions. Let $\xi(t)$, $-\infty < t < \infty$, stationary Gaussian process,

$$\xi(t)=\int e^{i\lambda t}\Phi(d\lambda) \qquad (1.19)$$

- its spectral representation and P - probability measure in σ-algebra $\mathfrak{A}(T)$, generated by values $\xi(t)=\xi(\omega, t)$ on Ω (where parameter t passes certain multitude T on real straight line). Let the average value of this stationary process equally to 0, and spectral measure is $F(d\lambda)$. Let P_1 - be another probability measure, relative to which stationary process with the zero average and spectral measure $F_1(d\lambda)$ form values

$$\xi_1(t)=\xi(t)-a(t),$$

where a (t) - the average value of initial values ξ (t):

$$a(t) = M_1\xi(t), \quad t \in T.$$

In the examination of Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}(T)$ of initial values (1.19) it is possible to pass to their linear combinations of the form

$$\eta(\varphi) = \int \varphi(\lambda)\,\Phi(d\lambda), \qquad (1.20)$$

where

$$\varphi(\lambda) = \sum_k c_k e^{i\lambda t_k}, \qquad (1.21)$$

$t_1, \ldots, t_n \in T$, a $c_1, \ldots, c_n$ - real coefficients.

Let us designate L_T^0 the linear space of all functions $\varphi(\lambda)$ form (1.21). Representable by formula (1.20) of value $\eta(\varphi),\ \varphi \in L_T^0,$ can be considered as Gaussian functional on space L_T^0. Let us assume

$$a(\varphi) = M_1\eta(\varphi), \quad \varphi \in L_T^0. \qquad (1.22)$$

Let us note that $a(\varphi) = \sum_k c_k a(t_k)$ with $\varphi(\lambda) = \sum_k c_k e^{i\lambda t_k}$.

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Let us consider L_T^0 as subspace hilbert space $L_T(F)$ with the scalar product

$$\langle\varphi, \psi\rangle_F = \int_{-\infty}^{\infty} \varphi(\lambda)\,\overline{\psi(\lambda)}\,F(d\lambda), \qquad (1.23)$$

where F (dλ) - the spectral measure of stationary process ξ (t) - relative to distribution P. Recall that $L_T(F)$ there is closing/shorting of space L_T^0 relative to scalar product (1.23). Simultaneously let us consider L_T^0 as subspace in hilbert space $L_T(F_1)$ with scalar product $\langle\varphi, \psi\rangle_{F_1}$ (here F_1 (dλ) - the spectral measure of stationary process $\xi_1(t)=\xi(t)-a(t)$ relative to distribution P_1).

The correlation function of assigned by formula (1.20) of random functional $\eta(\varphi)$ of the functional parameter $\varphi(\lambda)\in L_T^0$ is

$$B(\varphi, \psi)=\langle\varphi, \psi\rangle_F, \quad \varphi, \psi\in L_T^0, \tag{1.24}$$

relative to distribution P and

$$B_1(\varphi, \psi)=\langle\varphi, \psi\rangle_{F_1}, \quad \varphi, \psi\in L_T^0, \tag{1.25}$$

relative to distribution P_1.

It is clear that if $\|\varphi\|_F=0$, but $\|\varphi\|_{F_1}\neq 0$ for certain function $\varphi(\lambda)\in L_T^0$, then the measures P and P_1 are orthogonal, since

$$P\{\eta(\varphi)=0\}=1, \quad P_1\{\eta(\varphi)=0\}=0. \tag{1.26}$$

Moreover,

probability measures P and P_1 are orthogonal with the disturbance of the following condition [1]:

$$\|\varphi\|_F\asymp\|\varphi\|_{F_1}, \quad \varphi\in L_T^0. \tag{1.27}$$

FOOTNOTE [1]. Recall that the relationship/ratio $\|\varphi\|_F \asymp \|\varphi\|_{F_1}$ indicates $0 < c_1 \leqslant \leqslant \|\varphi\|_F / \|\varphi\|_{F_1} \leqslant c_2 < \infty$, where C_1 and C_2, some constants,. ENDFOOTNOTE.

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In fact, if, for example, there is a sequence $\varphi_n(\lambda) \in L_T^0$, $n = 1, 2, \ldots$, such, that

$$\|\varphi_n\|_F^2 = 1, \quad \sigma_n^2 = \|\varphi_n\|_{F_1}^2 \to 0 \quad \text{with} \quad n \to \infty,$$

that

$$\mathsf{P}\{|\eta(\varphi_n) - a(\varphi_n)| < \sqrt{\sigma_n}\} = \int\limits_{|x - a(\varphi_n)| < \sqrt{\sigma_n}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \to 0,$$

$$\mathsf{P}_1\{|\eta(\varphi_n) - a(\varphi_n)| < \sqrt{\sigma_n}\} = \int\limits_{|x| < \frac{1}{\sqrt{\sigma_n}}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \to 1.$$

Analogous relationship/ratios occur, also, in the case, when

$$\|\varphi_n\|^2_{F_1} = 1, \quad \|\varphi_n\|^2_F \to 0 \quad \textbf{with} \quad n \to \infty,$$

so that (see condition (1.5)) probability measures P and P_1 will be orthogonal.

Condition (1.27) indicates, in particular, that the hilbert spaces $L_T(F)$ and $L_T(F_1)$ coincide:

$$L_T(F) = L_T(F_1),$$

whereupon relationship/ratio (1.27) is fulfilled for all $\varphi(\lambda) \in L_T(F)$ $(\varphi \in L_T(F_1))$.

Further, let us consider average value (1.22) of random functional $\eta(\varphi)$, $\varphi \in L^0_T$, relative to probability measure P_1.

It is clear that if $a(\varphi) \neq 0$ with $\|\varphi\|_F = 0$, then the measures P and P_1 are orthogonal, since either $\|\varphi\|_{F_1} \neq 0$ they occur of relationship/ratio (1.26), or $\|\varphi\|_{F_1} = 0$ and then

$$\begin{aligned} \mathsf{P}\{\eta(\varphi) = a(\varphi)\} &= 0, \\ \mathsf{P}_1\{\eta(\varphi) = a(\varphi)\} &= 1. \end{aligned} \tag{1.28}$$

This indicates that for nonorthogonal measures the average value $a(\varphi)$, $\varphi \in L^0_T$, **is linear functional in hilbert space** $L_T(F)$.

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Let us show that for the nonorthogonal measures P and P_1 the average value $a(\varphi)$, $\varphi \in L_T^0$, is the limited linear functional in hilbert space $L_T(F)$.

In fact, it is possible to consider carried out condition (1.27), and then, if $a(\varphi_n) \to \infty$ with $n \to \infty$ for certain sequence $\varphi_n(\lambda) \in L_T^0$ such, that $\sigma_n = \|\varphi_n\|_{F_1} \asymp \|\varphi_n\|_F = 1$, then

$$\mathsf{P}\{\eta(\varphi_n) > \sqrt{a(\varphi_n)}\} = \frac{1}{\sqrt{2\pi}} \int\limits_{\sqrt{a(\varphi_n)}}^{\infty} e^{-x^2/2}\,dx \to 0,$$

$$\mathsf{P}_1\{\eta(\varphi_n) > \sqrt{a(\varphi_n)}\} = \frac{1}{\sqrt{2\pi}\sigma_n} \int\limits_{-a(\varphi_n)+\sqrt{a(\varphi_n)}}^{\infty} e^{-x^2/2\sigma_n^2}\,dx \to 1,$$

a this indicates the orthogonality of measures P and P_1 (see condition (1.5)).

Let $\phi_1, \phi_2, \ldots \in L_T^0$ - any sequence of functions, full/total/complete both in the hilbert space $L_T(F)$, and in hilbert space $L_T(F_1)$ (recall that subspace L_T^0 all functions of form (1.21) everywhere tightly both in $L_T(F)$, and in $L_T(F_1)$), and let $\mathfrak{A}$

indicates σ-algebra, generated by all values $\eta_k=\eta(\varphi_k)$, k = 1, 2, ..., form (1.20).

Lemma 1. The Gaussian measures P and P_1 are equivalent in σ-algebra $\mathfrak{A}(T)$ when and only when they are equivalent in σ-algebra $\mathfrak{A}\subseteq\mathfrak{A}(T)$.

Proof. Let P and P_1 be equivalent on $\mathfrak{A}$. Any cell/element φ (λ) $=e^{i\lambda t}$ in space $L_T(F)$ is a limit of the linear combinations of form $\psi_n(\lambda)=\sum_k c_{kn}\varphi_k(\lambda)$, and, therefore,

$$\mathsf{M}\,|\,\xi(t)-\eta(\psi_n)\,|^2=\|\,e^{i\lambda t}-\psi_n(\lambda)\,\|_F^2\to 0$$

with $n\to\infty$. For the equivalent measures P and P_1 carried out condition (1.27) and the average value a (φ) is linear continuous functional, so that

$$\mathsf{M}_1\,|\,\xi(t)-\eta(\psi_n)\,|^2=\|\,e^{i\lambda t}-\psi_n(\lambda)\,\|_{F_1}^2+|\,a(e^{i\lambda t})-a(\psi_n)\,|^2\to 0.$$

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Consequently, for fast enough increasing subsequence n_k, k = 1, 2, ...,

$$\xi(\omega,\ t)=\lim_{n\to\infty}\eta(\omega,\ \psi_{n_k})$$

almost everywhere both relative to the probability measure P and relative to P_1. It is obvious, values ξ (ω, t), determined by the asymptotic relation

$$\xi(\omega,\ t)=\lim_{k\to\infty}\eta(\omega,\ \psi_{n_k})$$

at those $\omega\in\Omega$, for which the limit indicated exists, and define by equality $\xi(\omega, t) = 0$ with the others $\omega\in\Omega$, will be equivalent to initial values $\xi(\omega, t)$ in the sense of condition (1.7). These values are measured relative to σ-algebra $\mathfrak{A}^*$, being addition/completion initial σ-algebra $\mathfrak{A}$ by the sets of probability 0 (relative to the equivalent on $\mathfrak{A}$ distributions P and P_1). It is obvious, the probability measures P and P_1 will be equivalent on $\mathfrak{A}^*$ and the fact more in σ-algebra $\mathfrak{A}_\xi$, generated by values $\xi(t)$, $t\in T$. Taking into account condition (1.7), hence we include that P and P_1 will be equivalent in σ-algebra $\mathfrak{A}(T)$. Q. E. D.

Let $\mathfrak{A}$ there is σ-algebra, generated by values $\eta_k(\omega)=\eta(\omega, \varphi_k)$ on Ω, where $\varphi_k(\lambda)$, k = 1, 2, ..., some set of functions from L_T^0, is full/total/complete in spaces $L_T(F)$ and $L_T(F_1)$. Let $\mathfrak{A}_n$ indicate σ-algebra, generated only by values η_k, k = 1, ..., n. It is obvious, $\mathfrak{A}$ is minimum σ-algebra, which contains entire sequence $\mathfrak{A}_1\subseteq\mathfrak{A}_2\subseteq$ ..., that conditionally can be registered as

$$\mathfrak{A} = \lim_{n\to\infty}\mathfrak{A}_n.$$

let us consider the Gaussian measures P and P_1, equivalent in σ-algebra $\mathfrak{A}$. Let

$$p(\omega) = \frac{P_1(d\omega)}{P(d\omega)}$$

there is their density on $\mathfrak{A}$.

It is obvious, the measures P and P_1 are equivalent in any σ-algebra $\mathfrak{A}'$, containing in $\mathfrak{A}$, whereupon for the appropriate density $p'(\omega) = P_1(d\omega)/P(d\omega)$ (measurable relatively $\mathfrak{A}'$)

$$P_1(A') = \int_{A'} p'(\omega) P(d\omega) = \int_{A'} p(\omega) P(d\omega),$$

whatever set $A' \in \mathfrak{A}'$. It is evident that density $p'(\omega)$ coincides with the conditional mathematical expectation of value $p(\omega)$ (relative to σ-algebra $\mathfrak{A}'$):

$$p'(\omega) = M\{p(\omega)/\mathfrak{A}'\}. \qquad (1.29)$$

According to the known properties of conditional mathematical expectations, [1]. for densities $p_n(\omega) = P_1(d\omega)/P(d\omega)$ in σ-algebra $\mathfrak{A}_n \subseteq \mathfrak{A}$, representable in the form

$$p_n(\omega) = M\{p(\omega)/\mathfrak{A}_n\}, \quad n = 1, 2, \ldots, \qquad (1.30)$$

we have: with probability 1

$$p(\omega) = \lim_{n\to\infty} p_n(\omega) \qquad (1.31)$$

(whereupon asymptotic relation (1.31) it is implemented also on the average).

FOOTNOTE [1]. See, for example, [12], page 287. ENDFOOTNOTE.

Further, let us turn to the arbitrary density p' (ω) form (1.29). Applying known inequality [2]. for the convex function log x, we have

$$\log p'(\omega) = \log \mathrm{M}\{p(\omega)/\mathfrak{A}'\} \geqslant \mathrm{M}\{\log p(\omega)/\mathfrak{A}'\}$$

and, therefore,

$$\mathrm{M}\log p' \geqslant \mathrm{M}\log p. \qquad (1.32)$$

FOOTNOTE [2]. See [12], page 37. ENDFOOTNOTE.

Accordingly common/general/total inequality, for the sequence of the densities of form (1.30) we obtain, that

$$\mathrm{M}\log p_1 \geqslant \mathrm{M}\log p_2 \geqslant \ldots$$

$$p_n \geqslant \mathrm{M}\log p,$$

Since M log ^ with M log p > - ∞ for the monotonic sequence M log p_n, n = 1, 2, ..., there is the final limit

$$\lim_{n\to\infty} \mathrm{M}\log p_n. \qquad (1.33)$$

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It will be shown below (see theorem 1), that is really/actually M log p > - ∞.

Thus, density p (ω) = P_1 (dω) /P (dω) in σ-algebra $\mathfrak{A}$ (but that means and in σ-algebra $\mathfrak{A}(T)$) it can be found from the asymptotic relation of the form

$$\log p(\omega) = \lim_{n\to\infty} \log p_n(\omega) = \\ = \lim_{n\to\infty} \mathrm{M} \log p_n(\omega) + \lim_{n\to\infty} [\log p_n(\omega) - \mathrm{M} \log p_n(\omega)], \quad (1.34)$$

that also it will be done subsequently.

§2. Some conditions of the equivalency of Gaussian measures.

1. Conditions of equivalency, connected with the entropy of distributions. Let us consider the Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}(T)$, generated by all values $\eta(\varphi) = \eta(\omega, \varphi)$ form (1.20) on probability space Ω, where the functional parameter φ (λ) passes linear space L_T^0. Let us assume that which was carried out condition (1.27), with disturbance/breakdown of which measure P and P_1 they are orthogonal.

Let $\eta_k = \eta(\varphi_k)$, k = 1, 2, ..., the sequence of the values of form (1.20), where $\varphi_1, \varphi_2, \ldots \in L_T^0$ - certain full/total/complete

system of linearly independent functions in each of the hilbert spaces $L_T(F)$ and $L_T(F_1)$, and let $\mathfrak{A}$ is σ-algebra, generated by all these values. On any of σ-algebra $\mathfrak{A}_n$, generated by values $\eta_k = \eta_k(\omega)$ to Ω (k = 1, ..., n) the Gaussian measures P and P_1 they are equivalent, since on the strength of predicted condition (1.27) the corresponding correlation matrix/dies { B (k, j) } and { B_1 (k, j) } are nondegenerate.

Let us consider the Gaussian measure P_1 in σ-algebra $\mathfrak{A}_n$, a formed by all multitude of the form

$$A = \{[\eta_1, \ldots, \eta_n] \in \Gamma\},$$

where Γ - the borel sets of N-dimensional vector space R^n. Page 110.

Let (a_1, ..., a_n) - the average value of relatively probability measure P_1, i.e., $a_k = M_1\eta(\varphi_k)$, k = 1, 2, ... Then

$$P_1(A) = \int_{R^n} f_1(x_1, \ldots, x_n)\, dx_1 \ldots dx_n,$$

where f_1 (x_1, ..., x_n) - the density of the corresponding Gaussian distribution in space R^n vectors x = (x_1, ..., x_n) - it takes the form

$$f_1(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2} D^{1/2}} \exp\left\{-\frac{1}{2} \sum (x_k - a_k)(x_j - a_j) C_1(k, j)\right\}$$

(D indicates the determinant of matrix/die { B_1 (k, j) }, and { C_1

(k, j) } - reciprocal matrix to { B_1 (k, j) }).

Let f (x_1, ..., x_n) - the density of the corresponding distribution P in vector space R^n (having the same form, as density f_1 (x_1, ..., x_n), but with zero average value and correlation matrix/die { B (k, j) }). Then density of distribution is

$$\frac{\mathsf{P}_1(dx)}{\mathsf{P}(dx)} = \frac{f_1(x_1, \ldots, x_n)}{f(x_1, \ldots, x_n)}, \qquad x \in R^n.$$

Consequently, the corresponding density $p_n(\omega) =$ P_1 (dω) /P (dω) on the initial space Ω is

$$p_n(\omega) = \frac{f_1[\eta_1(\omega), \ldots, \eta_n(\omega)]}{f[\eta_1(\omega), \ldots, \eta_n(\omega)]}$$

and can be described by the formula

$$\log p_n(\omega) \doteq \log \frac{D}{D_1} - \\ - \frac{1}{2} \sum_{j,k=1}^{n} [C_1(j, k)(\eta_j(\omega) - a_j)(\eta_k(\omega) - a_k) - \\ - C(j, k)\eta_j(\omega)\eta_k(\omega)], \qquad (2.1)$$

where D and D_1 they indicate the determinants of matrix/dies { B (j, k) } and { B_1 (j, k) }, k, j = 1, ..., n, but { C (j, k) } and { C_1 (j, k) } - the matrix/dies, reverse to { B (j, k) } and { B_1 (j, k) }.

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In generated cell/elements $\varphi_1, \ldots, \varphi_n$ subspace it is possible to select the base $\varphi_{1n}, \ldots, \varphi_{nn}$, in which both positive bilinear forms $B(\varphi, \psi)$ and $B_1(\varphi, \psi)$ are led to the diagonal form:

$$B(\varphi_{kn}, \varphi_{jn}) = \begin{cases} 1 & \text{with } k = j, \\ 0 & \cancel{\text{при}} \quad k \neq j; \end{cases}$$

$$B_1(\varphi_{kn}, \varphi_{jn}) = \begin{cases} \sigma_{kn}^2 & \cancel{\text{при}} \quad k = j, \\ 0 & \cancel{\text{при}} \quad k \neq j. \end{cases}$$

then density $p_n(\omega)$ = P_1 ($d\omega$) /P ($d\omega$) in σ-algebra $\mathfrak{A}_n$ can be described by the following formula (comp. (2.1)):

$$p_n(\omega) = \frac{1}{\prod_{k=1}^{n} \sigma_{kn}} \exp\left\{ -\frac{1}{2} \sum_{k=1}^{n} \left[\frac{(\eta_{kn}(\omega) - a_{kn})^2}{\sigma_{kn}^2} - \eta_{kn}^2(\omega) \right] \right\}, \tag{2.2}$$

where

$$\eta_{kn} = \eta(\varphi_{kn}), \quad a_{kn} = \mathsf{M}_1 \eta(\varphi_{kn}), \quad k = 1, \ldots, n.$$

It is easy to consider that

$$\begin{aligned} \mathsf{M} \log p_n &= \frac{1}{2} \sum_{k=1}^{n} \left(\log \frac{1}{\sigma_{kn}^2} - \frac{1}{\sigma_{kn}^2} + 1 - \frac{a_{kn}^2}{\sigma_{kn}^2} \right), \\ \mathsf{M}_1 \log p_n &= \frac{1}{2} \sum_{k=1}^{n} (-\log \sigma_{kn}^2 + \sigma_{kn}^2 - 1 + a_{kn}^2), \\ \mathsf{D} \log p_n &= \frac{1}{2} \sum_{k=1}^{n} \frac{(1 - \sigma_{kn}^2)^2 + 2a_{kn}^2}{\sigma_{kn}^4}, \\ \mathsf{D}_1 \log p_n &= \frac{1}{2} \sum_{k=1}^{n} [(1 - \sigma_{kn}^2)^2 + 2\sigma_{kn}^2 a_{kn}^2], \end{aligned} \tag{2.3}$$

where $D\eta$ and $D_1\eta$ indicate dispersions relative to P and P_1.

Let us define the so-called entropy distance $r(\mathfrak{A}')$ between the equivalent Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}' \subseteq \mathfrak{A}$ as

$$r(\mathfrak{A}') = -\left[\mathsf{M}\log\frac{\mathsf{P}_1(d\omega)}{\mathsf{P}(d\omega)} + \mathsf{M}_1\log\frac{\mathsf{P}(d\omega)}{\mathsf{P}_1(d\omega)}\right]. \quad (2.4)$$

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According to common/general/total inequality (1.32)

$$r(\mathfrak{A}') \leqslant r(\mathfrak{A}'') \text{ with } \mathfrak{A}' \subseteq \mathfrak{A}''. \quad (2.5)$$

Let us assume $r_n = r(\mathfrak{A}_n)$. Since $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \ldots$, the sequence r_n, n = 1, 2, ..., is monotonically increasing.

From formulas (2.3) it is evident that if

$$\inf_{k,n} \sigma_{kn}^2 = 0 \text{ or } \sup_{k,n} \sigma_{kn}^2 = \infty, \quad (2.6)$$

that

$$\lim_{n\to\infty} r_n = \infty. \quad (2.7)$$

But if

$$\sigma_{kn}^2 \asymp 1, \quad (2.8)$$

which, obviously, is equivalent to relationship/ratio (1.27), then

$$-\log\frac{1}{\sigma_{kn}^2}+\frac{1}{\sigma_{kn}^2}-1\asymp-\log\sigma_{kn}^2+\sigma_{kn}^2-1\asymp(1-\sigma_{kn}^2)^2,$$

and therefore

$$\mathsf{M}[-\log p_n]\asymp\mathsf{M}_1\log p_n\asymp\mathsf{D}\log p_n\asymp\mathsf{D}_1\log p_n\asymp$$
$$\asymp r_n\asymp\sum_{k=1}^{n}[(1-\sigma_{kn}^2)^2+a_{kn}^2]. \quad (2.9)$$

Lemma 2. If is fulfilled relationship/ratio (2.7), then the Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}$ are orthogonal.

Proof. The case, when is fulfilled relationship/ratio (2.6), is already examined earlier (see condition (1.27)). It was shown, that in this case the probability measures P and P_1 were orthogonal.

Let be is fulfilled relationship/ratio (2.8) (but together with it and (2.9)).

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Then for the sequence of sets $A_n\in\mathfrak{A}_n$ the form

$$A_n=\left\{\log p_n-\mathsf{M}\log p_n\geqslant\frac{1}{2}r_n\right\}=$$
$$=\Omega\setminus\left\{-\log p_n+\mathsf{M}_1\log p_n\geqslant\frac{1}{2}r_n\right\}$$

according to Chebyshev's inequality occur the following asymptotic relations:

$$P(A_n) \leqslant \frac{D \log p_n}{\frac{1}{4} r_n^2} \asymp \frac{1}{r_n} \to 0$$

and

$$P_1(A_n) = 1 - P_1(\Omega - A_n) \geqslant 1 - \frac{D_1 \log p_n}{\frac{1}{4} r_n^2} \asymp 1 - \frac{1}{r_n} \to 1.$$

This speaks about that (see condition (1.5)), that the probability measures P and P_1 are orthogonal. Lemma is demonstrated.

Lemma 3. If

$$\lim_{n \to \infty} r_n < \infty, \tag{2.10}$$

that the probability measures P and P_1 are equivalent in σ-algebra $\mathfrak{A}$.

Proof. Let us suppose for example that the measure P_1 is not absolutely continuous relative to P. This means that for certain set $A \in \mathfrak{A}$

$$P(A) = 0, \quad P_1(A) \neq 0.$$

It is obvious, there is a sequence of sets $A_n \in \mathfrak{A}_n$ such, that $P_2(A \circ A_n) \to 0$ with $n \to \infty$, where measure $P_2 = P + P_1$, and consequently, for such multitudes $A_n \in \mathfrak{A}$

$$P(A \circ A_n) \to 0, \quad P_1(A \circ A_n) \to 0.$$

Let us consider σ-algebra $\mathfrak{A}'_n$, besides void set and entire space containing only set A_n and its addition. Density $p'_n(\omega) = P_1$ (dω) /P (dω) on $\mathfrak{A}'_n$ is equal to $\frac{P_1(A_n)}{P(A_n)}$ with $\omega \in A_n$ and $\frac{1 - P_1(A_n)}{1 - P(A_n)}$ with $\omega \notin A_n$, and

$$M_1 \log p'_n = \log \frac{P_1(A_n)}{P(A_n)} P_1(A_n) + \log \frac{1 - P_1(A_n)}{1 - P(A_n)} [1 - P_1(A_n)].$$

Since $P_1(A_n) \to P_1(A) > 0$, a $P(A_n) \to P(A) = 0$, then $M_1 \log p'_n \to \infty$ with $n \to \infty$. On the strength of common/general/total inequality (1.32)

$$-M_1 \log p'_n \leqslant -M_1 \log p_n \leqslant r_n,$$

where, as before, $p_n(\omega) = P_1(d\omega)/P(d\omega)$ indicates the density in σ-algebra $\mathfrak{A}_n \supseteq \mathfrak{A}'_n$.

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Thus, if the measure P_1 is not absolutely continuous relative to P, then $r_n \to \infty$ with $n \to \infty$. Analogous reasonings lead to the same result, when the measure P is not absolutely continuous relative to P_1. Consequently, under condition (2.10) of measure P and P_1 are equivalent in σ-algebra $\mathfrak{A}$, Q. E. D.

The obvious corollary of demonstrated above lemmas 2 and 3 is the following result. [1].

FOOTNOTE [1]. This result is obtained by Hajek (J. Hajek, about one property of the normal distributions of arbitrary stochastic process, Czech. math. i. 8 (1958), 610-618). ENDFOOTNOTE.

Theorem 1. The Gaussian measures P and P_1 either are equivalent or orthogonal, whereupon they are equivalent in σ-algebra $\mathfrak{A}(T)$ when and only when is satisfied condition (2.10).

Let $P^0{}_1$ be a Gaussian measure with zero mathematical expectation and the same correlation function, as P_1. From theorem 1 and relationship/ratio (2.9) it is easy to deduce the following important fact.

Theorem 2. The Gaussian measures P and P_1 are equivalent only if are equivalent pairs P and $P^0{}_1$ and P_1, whereupon for the equivalent measures P_1^0, P_1^0 and P_1 density $P_1(d\omega)/P(d\omega)$ is

$$\frac{P_1(d\omega)}{P(d\omega)}=\frac{P_1(d\omega)}{P_1^0(d\omega)}\,\frac{P_1^0(d\omega)}{P(d\omega)}. \qquad (2.11)$$

(Recall that we examine the case, when $P = P^0$).

2. The conditions of equivalency, connected with hilbert spaces $L_T(F)$ and $L_T(F_1)$.

let P and P_1 - Gaussian measures in σ-algebra $\mathfrak{A}(T)$, that which was generated by all values $\eta(\varphi)$ form (1.20), where the functional

parameter ♦ (λ) passes space L_T^0.

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According to theorem 2 it is possible to examine separately two cases. The first - when the correlation functions B $(\varphi, \psi) = \langle\varphi, \psi\rangle_F$ and $B_1(\varphi, \psi) = \langle\varphi, \psi\rangle_{F_1}$ coincide:

$$\langle\varphi, \psi\rangle_F = \langle\varphi, \psi\rangle_{F_1} \qquad (2.12)$$

with all $\varphi, \psi \in L_T^0$. The second - when the average value $M_1\eta(\varphi)$ is equal to 0.

Let us consider the first case, when the Gaussian measures P and P_1 differ only in terms of the average value:

$$M\eta(\varphi) = 0, \quad M_1\eta(\varphi) = a(\varphi), \quad \varphi \in L_T^0.$$

theorem 3. Under condition (2.12) the Gaussian measures P and P_1 are equivalent only if the average value a (♦) is a linear continuous functional in hilbert space $L_T(F)$:

$$a(\varphi) = \langle\varphi, \psi\rangle_F, \quad \varphi \in L_T^0, \qquad (2.13)$$

where $\psi(\lambda)$ - the determined cell/element from $L_T(F)$.

proof. The continuity of the linear functional a (♦) in hilbert space $L_T(F)$ is equivalent the limitedness of this functional and for equivalent Gaussian measures it was establish/installed earlier (see Section 3 §1).

Let the average value a (ϕ) be continuous linear functional. KA any linear continuous functional, the average value a (ϕ), $\varphi \in L_T^0$, is described by formula (2.13), in which the cell/element $\psi(\lambda) \in L_T(F)$ is determined unambiguously, since the subspace L_T^0 is tight in $L_T(F)$. Let ϕ_1, ϕ_2, ... $\in L_T^0$ - the full/total/complete orthonormal set in $L_T(F)$. As can be seen from formulas (2.3), the entropy distance between the Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}_n$, generated by values $\eta_k = \eta(\varphi_k)$, k = 1, ..., n, is

$$r_n = 2\sum_{k=1}^{n} a_k^2$$

(where $a_k = a(\varphi_k)$) and

$$\lim_{n\to\infty} r_n = 2\sum_{k=1}^{\infty} a_k^2 = 2\sum_{k=1}^{\infty} \langle\varphi_k, \psi\rangle_F^2 = 2\|\psi\|_F^2 < \infty.$$

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Thus, is satisfied condition (2.10), and, consequently (see theorem 1), the Gaussian measures P and P_1 are equivalent in σ-algebra $\mathfrak{A} = \lim \mathfrak{A}_n$. On lemma 1 this is equivalent equivalency P and P_1 in σ-algebra $\mathfrak{A}(T)$. Theorem is demonstrated.

Let us consiler now the Gaussian measures P and P_1 (in σ-algebra $\mathfrak{A}(T)$) with identical, equal to 0, by the average values.

Let us determine operator A from hilbert spaces $L_T(F)$ into having hilbert space $L_T(F_1)$, placed

$$A\varphi(\lambda) = \varphi(\lambda) \qquad (2.14)$$

for all $\varphi(\lambda) \in L_T^0$. As earlier, let us assume that which was carried out condition (1.27), with disturbance/breakdown of which the measures P and P_1 are orthogonal. This condition is equivalent to the fact that the operator A is limited and has the bounded inverse operator; this condition can be registered also in the form

$$A^*A \asymp E, \qquad (2.15)$$

where A * is the adjoint operator to A, E - single operator, and relationship/ratio (2.15) means that

$$\|(A^*A)\varphi\|_F \asymp \|\varphi\|_F, \quad \varphi \in L_T(F).$$

Let us note for a clarity that

$$\langle A^*A\varphi, \psi\rangle_F = \langle A\varphi, A\psi\rangle_{F_1} = \langle \varphi, \psi\rangle_{F_1} \qquad (2.16)$$

with any φ, ψ.

Let us consider the difference

$$\Delta = E - A^*A. \qquad (2.17)$$

Lemma 4. If the operator Δ is completely continuous, then condition (2.15), and together with it and condition (1.27), is satisfied when and only when operator Δ does not have equal to 1 eigenvalue.

Proof. It is obvious, condition (2.15) is equivalent to the fact that operator A*A is limited and has the bounded inverse operator $(A^*A)^{-1}$.

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Further, since the operator A*A is positive, difference Δ = E - A*A is such, that

$$\delta = \sup_{\|\varphi\|_F = 1} \langle \Delta\varphi, \varphi\rangle \leqslant 1.$$

We have

$$\langle \varphi, \varphi\rangle_F - \langle A^*A\varphi, \varphi\rangle_F \leqslant \delta \langle \varphi, \varphi\rangle_F,$$
$$\langle A^*A\varphi, \varphi\rangle_F \geqslant (1-\delta) \langle \varphi, \varphi\rangle_F.$$

Consequently, with $\delta \neq 1$ there is a bounded operator $(A^*A)^{-1}$. On the other hand, if 1 is the eigenvalue cf operator E - A*A, then 0 are the eigenvalue of operator A*A, and therefore inverse operator $(A^*A)^{-1}$ does not exist.

Theorem 4. Under condition (1.27) the Gaussian measures P and P_1 are equivalent when and only when Δ = E - A*A is the operator of gil'berta - Schmilt. [1].

FOOTNOTE [1]. Recall here, that the operator gilbert-of Schmidt is called the completely continuous operator, the full/total/complete system of eigenvalues μ_1, μ_2, ... whom it satisfies condition $\sum_k \mu_k^2 < \infty$. Theorem 4 is obtained for the first time Feldman (J Feldman, Equivalence and perpendicularity of Gaussian processes, Pacif. J Math. 8 (1958) 699-708; 9 (1959) 1295-1296); we it is easily concluded from theorem 1, whereas the proof of Feldman is very complicated. ENDFOOTNOTE.

Proof. Let us consider spectral representation of the bounded symmetrical operator Δ:

$$\Delta = \int \mu E(d\mu),$$

where $E(d\mu)$ - the spectral family of projection operators ("resolution of unity"). It is clear that

$$A^*A = \int (1-\mu) E(d\mu).$$

Let us assume that operator's spectrum Δ is not purely discrete. Then, obviously, outside certain vicinity $(-\varepsilon, \varepsilon)$ is an infinite number of points of the spectrum and, consequently, also an infinite number of nonintersecting intervals $[\mu_k, \mu_{k+1}]$, $k = 1, 2, \ldots$, such, that the invariant orthogonal subspaces of the form

$$E[\mu_k, \mu_{k+1}] L_T(F)$$

continually are different from zero.

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Let us select in each of these subspaces on equivalent component $\varphi_k, \|\varphi_k\|_F = 1$. For them

$$\langle A^*A\varphi_k, \varphi_j\rangle_F = \langle\varphi_k, \varphi_j\rangle_{F_1} = \begin{cases} \sigma_k^2 & \text{with } j = k, \\ 0 & \text{with } j \neq k, \end{cases}$$

whereupon

$$\mu_k \leqslant 1 - \sigma_k^2 \leqslant \mu_{k+1}, \quad (1 - \sigma_k^2)^2 \geqslant \varepsilon^2.$$

Entropy distance r_n between the Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}_n$, generated by values $\eta_k = \eta(\varphi_k)$, k = 1, ..., n, it is such, that (see (2.9))

$$r_n \asymp \sum_{k=1}^{n} (1 - \sigma_k^2)^2 \geqslant \varepsilon^2 n.$$

It is evident that $r_n \to \infty$ with $n \to \infty$, and according to theorem 1 Gaussian measures P and P_1 are orthogonal in σ-algebra $\mathfrak{A} = \lim_{n\to\infty} \mathfrak{A}_n$.

Thus, for the equivalent measures P and P_1 operator's spectrum Δ (but that means and operator A*A) it is purely discrete. If $\phi_1, \phi_2,$ ... - the full/total/complete orthonormalized set of functions with eigenvalues $\mu_1, \mu_2, \ldots,$ then the condition

$$\sum_k \mu_k^2 < \infty \qquad (2.18)$$

is equivalent to the fact that

$$\lim_{n\to\infty} r_n \asymp \sum_{k=1}^{\infty}(1-\sigma_k^2)^2 < \infty,$$

where $\sigma_k^2 = 1 - \mu_k$, k = 1, 2, ..., the full/total/complete system of the eigenvalues of operator A*A, a r_n is an entropy distance between P and P_1 in σ-algebra $\mathfrak{A}_n$, generated by values $\eta_k = \eta(\varphi_k)$, k = 1, ..., n. Thus, condition (2.18) (together with the predicted condition (1.27)) is necessary and sufficient for the equivalency of the Gaussian measures P and P_1, Q. E. D.

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Let us note that condition (2.18) can be registered in the form

$$\sum_{k,l} \langle \Delta\varphi_k, \varphi_l\rangle_F^2 < \infty$$

and that for any orthonormal set $\psi_1, \psi_2, \ldots \in L_T(F)$

$$\sum_{k,l}\langle\Delta\psi_k, \psi_l\rangle_F^2 = \sum_k\Big[\sum_l\langle\Delta\psi_k, \psi_l\rangle_F\Big]^2 = \\ = \sum_k \|\Delta\psi_k\|_F^2 = \sum_k\Big[\sum_l\langle\Delta\psi_k, \varphi_l\rangle_F^2\Big] = \\ = \sum_l\Big[\sum_k\langle\psi_k, \Delta\varphi_l\rangle_F^2\Big] \leqslant \sum_l\|\Delta\varphi_l\|_F^2 = \sum_{k,l}\langle\Delta\varphi_k, \varphi_l\rangle_F^2,$$

whereupon for the full/total/complete orthonormal set $\psi_1, \psi_2, \ldots$ in these relationship/ratios the corresponding inequalities are converted into precise equalities.

It is easy to see that the operator Δ is the operator gilbert-of Schmidt (occurs discrete spectrum, and the full/total/complete system of eigenvalues satisfies conditions (2.18)) when and only when for any full/total/complete orthonormal set $\phi_1, \phi_2, \ldots$

$$\sum_{k, l} \langle \Delta\varphi_k, \varphi_l \rangle_F^2 < \infty. \qquad (2.19)$$

This condition can be directly expressed by the correlation functionals of distributions P and P_1, since

$$\langle \Delta\varphi, \psi \rangle_F = \langle \varphi, \psi \rangle_F - \langle A^* A\varphi, \psi \rangle_F = \\ = \langle \varphi, \psi \rangle_F - \langle \varphi, \psi \rangle_{F_1} = B(\varphi, \psi) - B_1(\varphi, \psi)$$

for any $\varphi, \psi \in L_T(F)$. Thus, under condition (1.27) for the equivalency of the Gaussian measures P and P_1 it is necessary and sufficiently, in order to for any full/total/complete orthonormal set $\phi_1, \phi_2, \ldots, \in L_T(F)$

$$\sum_{k, l} b(\varphi_k, \varphi_l)^2 < \infty, \qquad (2.20)$$

where

$$b(\varphi, \psi) = B(\varphi, \psi) - B_1(\varphi, \psi), \quad \varphi, \psi \in L_T(F).$$

Example. Let T = (-∞, ∞) there is entire/all real straight line.

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In this case $L_T(F)$ there is a set of all functions $\phi(\lambda)$ such, that $\int |\varphi(\lambda)|^2 F(d\lambda) < \infty$. Condition (1.27) (indicating, in particular, which $\int\limits_\Delta F(d\lambda) \asymp$

$\asymp \int\limits_{\Delta} F_1(d\lambda))$ is equivalent to the fact that the spectral measures F and F_1 are equivalent, and their density f (λ) = F_1 $(d\lambda)$ /F $(d\lambda)$ satisfies condition $f(\lambda) \asymp 1$ (with almost all λ). Operator A*A easily is located from the equality

$$\langle A^*A\varphi,\ \psi\rangle_F = \langle\varphi,\ \psi\rangle_{F_1} = \langle f\varphi,\ \psi\rangle_F,\ \varphi,\ \psi \in L_T(F).$$

Specifically,, since $f\varphi \in L_T(F)$, that A*A there is an operator of multiplication by density f (λ). Consequently, operator Δ = E - A*A is an operator of multiplication by function 1 - f (λ). It is clear that the function ϕ (λ) is its own for an operator Δ when and only when ϕ (λ) = 0 almost everywhere outside set { λ: 1 - f (λ) = μ }, where μ is the corresponding eigenvalue. It is easy to see that Δ is the operator gilbert-of Schmidt when and only when 1 - f (λ) = 0 almost everywhere relative to the continuous part of the spectral measure F $(d\lambda)$ (i.e. the continuous parts of the spectral measures F and F_1 coincide), for the discrete part, concentrated at points λ_1, λ_2, ..., is satisfied the condition

$$\sum_k \left[1 - \frac{F_1(\lambda_k)}{F(\lambda_k)}\right]^2 < \infty,$$

where

$$\mu_k = 1 - f(\lambda_k) = 1 - \frac{F_1(\lambda_k)}{F(\lambda_k)},\quad k = 1,\ 2,\ \ldots,$$

is a full/total/complete system of the nonzero eigenvalues, which correspond to the eigenfunctions of the form

$$\varphi_k(\lambda)=\begin{cases}0 & \textbf{with} \quad \lambda\neq\lambda_k \\ 1 & \quad " \quad \lambda=\lambda_k\end{cases}, \quad k=1, 2, \ldots$$

(see condition (2.18)).

Let us pause in greater detail at condition (2.20) and the escape/ensuing from it corollaries.

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Let us consider hilbert space $L_{T\times T}(F\times F)$, analogous to the introduced previously space $L_T(F)$. Specifically,, let us consider linear space $L^0_{T\times T}$ all functions of the form

$$\varphi(\lambda, \mu)=\sum_{k, l} c_{kl} e^{i(\lambda s_k - \mu t_l)} \qquad (2.21)$$

(where $s_k, t_l \in T$ and c_{kl} - real coefficients) and let us define $L_{T\times T}(F\times F)$ as the hilbert space, which is obtained by closing/shorting $L^0_{T\times T}$ according to the scalar product

$$\langle\varphi, \psi\rangle_{F\times F}=\int\int \varphi(\lambda, \mu)\overline{\psi(\lambda, \mu)}\, F(d\lambda)\, F(d\mu). \qquad (2.22)$$

Obviously, if $\varphi', \varphi'' \in L_T(F)$, that

$$\varphi(\lambda, \mu)=\varphi'(\lambda)\overline{\varphi''(\mu)} \qquad (2.23)$$

entering in space $L_{T\times T}(F\times F)$, the system of all functions of form (2.23) it is full/total/complete in $L_{T\times T}(F\times F)$. This immediately escape/ensues from the determination of the hilbert spaces $L_T(F)$ and in question $L_{T\times T}(F\times F)$, if one considers that the functions $\varphi(\lambda, \mu)$

form (2.23) enter in $L_{T\times T}(F\times F)$, when ϕ' and ϕ'' they are functions of type (1.21), and

$$\|\varphi(\lambda,\mu)-\psi(\lambda,\mu)\|^2_{F\times F}=$$
$$=\int\int|\varphi'(\lambda)\overline{\varphi''(\mu)}-\psi'(\lambda)\overline{\psi''(\mu)}|^2 F(d\lambda)F(d\mu)\leqslant$$
$$\leqslant 2\int\int[|\varphi''(\mu)|^2|\varphi'(\lambda)-\psi'(\lambda)|^2+|\psi'(\lambda)|^2|\varphi''(\mu)-\psi''(\mu)|^2]\times$$
$$\times F(d\lambda)F(d\mu)=2\left[\|\varphi''\|^2_F\|\varphi'-\psi'\|^2_F+\|\psi'\|^2_F\|\varphi''-\psi''\|^2_F\right]$$

for any functions ϕ', ϕ'' and ψ', ψ'' the mentioned type.

Let $\phi_1, \phi_2, \ldots$ - any orthonormal set, in hilbert space $L_T(F)$.

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Then, obviously, the functions

$$\varphi_{kj}(\lambda,\mu)=\varphi_k(\lambda)\overline{\varphi_j(\mu)},\quad k,j=1,2,\ldots,$$

form the full/total/complete orthonormal set in hilbert space $L_{T\times T}(F\times F)$.

Under condition (2.20) let us assume

$$b_{kj}=b(\varphi_k,\varphi_j),\quad k,j=1,2,\ldots,$$

and let us define cell/element $\psi_0(\lambda,\mu)\in L_{T\times T}(F\times F)$ as

$$\psi_0(\lambda,\mu)=\sum_{k,j}b_{kj}\varphi_{kj}(\lambda,\mu). \qquad (2.24)$$

Formula (2.24) gives the expansion of cell/element $\psi_0(\lambda,\mu)$ in terms of orthonormal set $\varphi_{kj}(\lambda,\mu)$, so that

$$b(\varphi_k,\varphi_j)=\langle\varphi_k\overline{\varphi}_j,\psi_0\rangle_{F\times F}.$$

Obviously, this relationship/ratio extends to any linear combinations

$$\varphi'(\lambda)=\sum_k c'_k\varphi_k(\lambda),\quad \varphi''(\mu)=\sum_l c''_l\varphi_l(\mu),$$

so that

$$b(\varphi',\varphi'')=\langle\varphi'\overline{\varphi''},\psi_0\rangle_{F\times F}. \qquad (2.25)$$

by passage to the limit it is possible to pass in relationship/ratio (2.25) to arbitrary functions $\varphi',\varphi''\in L_T(F)$.

On the other hand, if a difference in the correlation functionals b (ϕ', ϕ''), ϕ', ϕ'' $\in L_T(F)$, allow/assumes representation of the form (2.25), then for any orthonormal set ϕ_1, ϕ_2, ...

$$\sum_{k,l} b(\varphi_k,\varphi_l)\leqslant\|\psi_0\|^2_{F\times F},$$

so that is satisfied condition (2.20).

Formula (2.25) means that the bilinear functional b (ϕ', ϕ'') to $L_T(F)$, that which is considered as functional during the appropriate functions $\varphi'(\lambda)\overline{\varphi''(\mu)}\in L_{T\times T}(F\times F)$, is continued to linear continuous functional in hilbert space $L_{T\times T}(F\times F)$.

We come to the following result, to some degree analogous to theorem 3.

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Theorem 5. Under condition (1.27) for the equivalency of the Gaussian measures P and P_1 it is necessary and sufficiently in order that the difference

$$b(\varphi', \varphi'') = B(\varphi', \varphi'') - B_1(\varphi', \varphi''),$$

as functional during the appropriate functions $\varphi'(\lambda)\overline{\varphi''(\mu)} \in L_{T\times T}(F\times F)$, would be continued to linear continuous functional on hilbert space $L_{T\times T}(F\times F)$.

Let us turn to hilbert space $L_{T\times T}(F\times F_1)$, determined analogously $L_{T\times T}(F\times F)$, with the scalar product

$$\langle\varphi, \psi\rangle_{F\times F_1} = \int\int \varphi(\lambda, \mu)\overline{\psi(\lambda, \mu)}\, F(d\lambda)\, F_1(d\mu)$$

comp. (2.22)). Is easy to see, [1]. that under condition (1.27)

$$\|\varphi\|^2_{F\times F} \asymp \|\varphi\|^2_{F\times F_1} \qquad (2.26)$$

FOOTNOTE [1]. It is useful to note what $L_{T\times T}(F\times F)$ is functional model for the hilbert space of values $\eta(\omega', \omega'')$ on the product Ω x Ω, which is the closing/shorting of the values of the form

$$\eta(\omega', \omega'') = \sum_{k,l} c_{kl}\xi(\omega', s_k)\,\xi(\omega'', t_l)$$

on the scalar product of the form

$$\int\int \eta_1(\omega', \omega'')\,\eta_2(\omega', \omega'')\, P(d\omega')\times P(d\omega'')$$

[redacted].

Specifically,, to values $\underset{\eta(\omega', \omega'')}{\lambda}$ indicated they correspond to function ϕ (λ, μ) form (2.21) and

$$\int\int \eta_1(\omega', \omega'')\,\eta_2(\omega', \omega'')\,P(d\omega')\times P(d\omega'') = (\varphi_1, \varphi_2)_{F\times F}.$$

It is analogous

$$\int\int \eta_1(\omega', \omega'')\,\eta_2(\omega', \omega'')\,P(d\omega')\times P_1(d\omega'') = (\varphi_1, \varphi_2)_{F\times F_1}.$$

Condition (1.27) means that

$$\int [\eta(\omega)]^2\,P(d\omega) \asymp \int [\eta(\omega)]^2\,P_1(d\omega)$$

for any values of form $\eta(\omega) = \sum_k c_k \xi(\omega, t_k)$.

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Therefore under condition (1.27)

$$\|\varphi\|^2_{F\times F} = \int P(d\omega'')\int \left|\sum_k \left[\sum_j c_{kj}\xi(\omega'', t_j)\right]\xi(\omega', s_k)\right|^2 P(d\omega') \asymp$$
$$\asymp \int P(d\omega'')\int \left|\sum_k \left[\sum_j c_{kj}\xi(\omega'', t_j)\right]\xi(\omega', s_k)\right|^2 P_1(d\omega') = \|\varphi\|^2_{F\times F_1}.$$

End FootNote

with $\varphi\in L^0_{T\times T}$. This relationship/ratio shows that the hilbert spaces $L_{T\times T}(F\times F)$ and $L_{T\times T}(F\times F_1)$ coincide:

$$L_{T\times T}(F\times F) = L_{T\times T}(F\times F_1),$$

whereupon relationship/ratio itself (2.26) is correct with all

$$\varphi\in L_{T\times T}(F\times F) \quad (\varphi\in L_{T\times T}(F\times F_1)).$$

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GAUSSIAN RANDOM PROCESSES. PART 1,(U)

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Obviously, under condition (2.26) functional b (ϕ', ϕ'') of form (2.25) is continued into linear continuous functional on hilbert space $L_{T\times T}(F\times F_1)$ and let us present in the form

$$b(\varphi',\varphi'')=\langle\varphi'\overline{\varphi''},\ \psi\rangle_{F\times F_1}, \qquad (2.27)$$

where $\psi(\lambda,\mu)$ - the determined cell/element from $L_{T\times T}(F\times F_1)$.

Analogously under condition (2.26) from representation (2.27) escape/ensues the representability of functional b (ϕ', ϕ'') by formula (2.25). But from relationship/ratio (2.27) escape/ensues also condition itself (1.27), but together with it and condition (2.26). Specifically,, since for any cell/element $\varphi\in L_T^0$

$$b(\varphi,\varphi)=\|\varphi\|_F^2-\|\varphi\|_{F_1}^2=\langle\varphi(\lambda)\overline{\varphi(\mu)},\ \psi\rangle,$$

that

$$\left|\|\varphi\|_F^2-\|\varphi\|_{F_1}^2\right|\leqslant\|\varphi(\lambda)\overline{\varphi(\mu)}\|_{F\times F_1}\|\psi\|_{F\times F_1}=$$
$$=\|\varphi\|_F\|\varphi\|_{F_1}\|\psi\|_{F\times F_1}.$$

It is evident that if $\|\varphi\|_F\to 0$, then also $\|\varphi\|_{F_1}\to 0$, and vice versa. Consequently, the introduced previously operator A from $L_T(F)$ in $L_T(F_1)$ is continuous and has the continuous inverse operator A^{-1}. But this (see (2.15)) it is equivalent to condition (1.27).

Thus, if in theorem 5 space $L_{T\times T}(F\times F)$ to replace by $L_{T\times T}$(F x F_1), then further condition (1.27) is made automatically. It is more

precise, occurs the following assumption. [1].

FOOTNOTE [1]. In another form theorem 6 and simpler theorem 2 were proposed by Parzen (for example, see E Parzen, Probability density functionals and reproducing kernel Hilbert spaces, "Time series analysis", ed. M Rosenblatt, N.J., 1963). ENDFOOTNOTE.

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Theorem 6. For the equivalency of the Gaussian measures P and P_1 it is necessary and sufficiently in order that the difference

$$b(\varphi', \varphi'') = B(\varphi', \varphi'') - B_1(\varphi', \varphi''),$$

as functional during functions $\varphi'(\lambda)\overline{\varphi''(\mu)} \in L^0_{T\times T}$, would be continued to linear continuous functional on hilbert space $L_{T\times T}(F \times F_1)$.

§3. General conditions of equivalency and formula for the density of equivalent distributions.

Let P and P_1 be Gaussian measures in σ-algebra $\mathfrak{A}(T)$, generated by all values $\xi(t)$, $t \in T$, stationary (with respect to measure P) random

process ξ (t); T is an arbitrary multitude on real straight line.

Let us consider first the case, when these measures differ only in terms of the average value

$$a(t) = \mathsf{M}_1 \xi(t)$$

(more precise, when P°_1 = P - see theorem 2).

Theorem 7. The Gaussian measures P and P_1 are equivalent in σ-algebra $\mathfrak{A}(T)$ when and only when the average value a (t) allow/assumes spectral representation of the form

$$a(t) = \int e^{-i\lambda t} \psi(\lambda) F(d\lambda) \qquad (3.1)$$

with $t \in T$, where $\psi(\lambda)$ is certain function, which satisfies condition $\int |\psi(\lambda)|^2 F(d\lambda) < \infty$. For equivalent measures integral equation (3.1) has solution $\psi(\lambda) \in L_T(F)$ and density p (ω) = P_1 (dω) /P (dω) can be described by the formula

$$p(\omega) = D \exp\left\{\int \psi(\lambda) \Phi(d\lambda)\right\}, \qquad (3.2)$$

where $\psi(\lambda)$ - the solution to equation (3.1) from space $L_T(F)$, a D - normalizing factor, D = $e^{-\frac{1}{2}\|\psi\|_F^2}$.

(Here, as before $F(d\lambda)$ - spectral measure, $\Phi(d\lambda)$ - the stochastic spectral measure of stationary process ξ (t)).

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Proof. Accorling to theorem 3 for a equivalency P and P_1 it is necessary and sufficiently in order that would occur representation (3.1) with function $\psi(\lambda)$ from Hilbert space $L_T(F)$, since on the strength of the completeness of set of functions $\varphi(\lambda)=e^{i\lambda t}$, $t\in T$, this is equivalent to relationship/ratio (2.13). Further, if relationship/ratio (3.1) occurs for any function $\psi(\lambda)$ from Hilbert space $L_{(-\infty,\infty)}(F)$ all integrated squared functions, then it occurs also for certain function $\psi_0(\lambda)$ from space $L_T(F)$ - the projection of cell/element $\psi(\lambda)$ on subspace $L_T(F)\subseteq L_{(-\infty,\infty)}(F)$, since

$$\langle e^{i\lambda t}, \psi(\lambda)\rangle_F = \langle e^{i\lambda t}, \psi_0(\lambda)\rangle_F$$

with all $t\in T$ (when $e^{i\lambda t}\in L_T(F)$).

Let us consider now equivalent measures P and P_1 and let us turn to their density p (ω) = $P_1(d\omega)/P(d\omega)$. Let us select the full/total/complete orthonormalized set of functions $\phi_1(\lambda)$, $\phi_2(\lambda)$, $\ldots\in L_T(F)$ and let us consider densities $p_n(\omega)=P_1(d\omega)/P(d\omega)$ in σ-algebra $\mathfrak{A}_n$, each of which is generated by the appropriate values $\eta_k=\eta(\varphi_k)$, k = 1, ..., n, form (1.20). As has already been indicated previously see p.

3 §1 of this chapter),

$$p(\omega)=\lim_{n\to\infty} p_n(\omega).$$

Obviously (see formula (2.2)),

$$p_n(\omega)=\exp\left\{\sum_{k=1}^{n} a_k\eta_k(\omega)-\frac{1}{2}\sum_{k=1}^{n} a_k^2\right\}=$$
$$=\exp\left\{\int\psi_n(\lambda)\,\Phi(d\lambda)-\frac{1}{2}\|\psi_n\|_F^2\right\},$$

where

$$a_k=\mathsf{M}_1\eta_k=\langle\varphi_k,\ \psi\rangle_F,\qquad k=1,\ 2,\ \ldots,$$

$\psi(\lambda)\in L_T(F)$- function from (3.1) and

$$\psi_n(\lambda)=\sum_{k=1}^{n} a_k\varphi_k(\lambda).$$

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We have

$$\lim_{n\to\infty}\psi_n(\lambda)=\lim\sum_{k=1}^{n}\langle\psi,\ \varphi_k\rangle_F\varphi_k(\lambda)=\psi(\lambda)$$

and

$$\lim_{n\to\infty}\|\psi_n(\lambda)\|_n^2=\lim\sum_{k=1}^{n} a_k^2=\|\psi(\lambda)\|_F^2.$$

Thus,

$$p(\omega)=\lim_{n\to\infty} p_n(\omega)=\exp\left\{\int\psi(\lambda)\,\Phi(d\lambda)-\frac{1}{2}\|\psi(\lambda)\|_F^2\right\},$$

Q. E. D.

Let us now move on to Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}(T)$ with zero average values and arbitrary correlation functions.

Let us turn to space H_2 (t), together with other spaces $H_n(T)$ introduced in §5 of chapter 1. This space H_2 (T) is the locked linear closure of all values $\xi(s)\xi(t)-B(s, t),\ s,\ t\in T$.

Let us consider everywhere the dense in H_2 (T) linear space of all (represented in symmetrical form) values

$$\eta=\sum_{k,j} c_{kj}[\xi(t_k)\xi(t_j)-B(t_k, t_j)] \qquad (3.3)$$

(with symmetrical real coefficients $c_{kj}=c_{jk},\ k,\ j=1,\ 2,\ \ldots)$ - the linear closure of values ξ (s) ξ (t) - B (s,t). Taking into account formula for torque/moments (see formula (5.10) chapter I)

$$M\xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)=$$
$$=B(t_1, t_2)B(t_3, t_4)+B(t_1, t_3)B(t_2, t_4)+B(t_1, t_4)B(t_2, t_3),$$

easy to count, that

$$M\eta'\eta''=\sum_{k,j}\sum_{m,n}c'_{kj}c''_{mn}B(t_k, t_m)B(t_j, t_n)+$$
$$+\sum_{k,j}\sum_{m,n}c'_{kj}c''_{mn}B(t_k, t_n)B(t_j, t_m)=$$
$$=2\sum_{k,j}\sum_{m,n}c'_{kj}c''_{mn}B(t_k, t_m)B(t_j, t_n) \qquad (3.4)$$

for any values

$$\eta' = \sum_{k,l} c'_{kl}[\xi(t_k)\xi(t_l) - B(t_k, t_l)]$$

and

$$\eta'' = \sum_{k,l} c''_{kl}[\xi(t_k)\xi(t_l) - B(t_k, t_l)]$$

the form in question.

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If we introduce stochastic measure $\Psi(d\lambda, d\mu)$, after placing

$$\Psi(\Delta_1 \times \Delta_2) = \Phi(\Delta_1)\overline{\Phi(\Delta_2)} - F(\Delta_1 \cap \Delta_2), \qquad (3.5)$$

that easy to see that each of the values of form (3.3) can be represented the formula

$$\eta = \int\int \varphi(\lambda, \mu)\Psi(d\lambda, d\mu), \qquad (3.6)$$

where

$$\varphi(\lambda, \mu) = \sum_{k,l} c_{kl} e^{i(\lambda t_k - \mu t_l)}. \qquad (3.7)$$

From formula (3.4) we obtain, that

$$M\eta'\eta'' = 2\int\int \varphi'(\lambda, \mu)\overline{\varphi''(\lambda, \mu)}\, F(d\lambda)\, F(d\mu) = 2\langle\varphi', \varphi''\rangle_{F\times F}, \tag{3.8}$$

where

$$\varphi'(\lambda, \mu) = \sum_{k,l} c'_{kl} e^{i(\lambda t_k - \mu t_l)}$$

and

$$\varphi''(\lambda, \mu) = \sum_{k,l} c''_{kl} e^{i(\lambda t_k - \mu t_l)}.$$

From relationship/ratios (3.6) - (3.8) it is evident that to convergent series $\eta_1, \eta_2, \ldots \in H_2$ corresponds by formula (3.6) the convergent series of functions $\varphi_1, \varphi_2, \ldots \in L_{T\times T}(F\times F)$. Thus, any value $\eta \in H_2$, as limit of certain sequence of values $\eta_1, \eta_2, \ldots$ form (3.3), represented by formula (3.6), where function $\varphi(\lambda, \mu) \in L_{T\times T}(F\times F)$ is the limit of the corresponding functions ϕ_1, ϕ_2, ... form (3.7), and for any this function $\varphi(\lambda, \mu)$ formula (3.6) determines certain value $\eta \in H_2(T)$.

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Theorem 8. Gaussian measures P and P_1 (with zero average

values) are equivalent in σ-algebra $\mathfrak{A}(T)$ when and only when a difference correlation the functions

$$b(s, t) = B(s, t) - B_1(s, t)$$

represented in the form

$$b(s, t) = \int\int e^{-i(\lambda s - \mu t)} \psi(\lambda, \mu) F(d\lambda) F_1(d\mu) \qquad (3.9)$$

with $s, t \in T$, where the function $\psi(\lambda, \mu)$ is such, that

$$\int\int |\psi(\lambda, \mu)|^2 F(d\lambda) F_1(d\mu) < \infty.$$

For equivalent measures P and P_1 integral equation (3.9) has solution $\psi(\lambda, \mu) \in L_{T\times T}(F \times F_1)$. Density $p(\omega) = \mathsf{P}_1(d\omega)/\mathsf{P}(d\omega)$ equivalent measures can be described by the formula

$$p(\omega) = D \exp\left\{-\frac{1}{2}\int\int \psi(\lambda, \mu) \Psi(d\lambda, d\mu)\right\}, \qquad (3.10)$$

where $\psi(\lambda, \mu)$ - the solution to equation (3.9) from space $L_{T\times T}(F \times F_1)$, and D - normalizing factor.

Proof. Since the functions $\varphi(\lambda, \mu) = e^{i(\lambda s - \mu t)}$, $s, t \in T$, form full/total/complete system in hilbert space $L_{T\times T}(F \times F_1)$, relationship (3.9) with $\psi(\lambda, \mu) \in L_{T\times T}(F \times F_1)$ is equivalent to relationship/ratio (2.27), which, according to theorem 6, indicates the equivalency of Gaussian measures P and P_1. But if relationship/ratio (3.9) occurs for certain function $\psi(\lambda, \mu)$ from

hilbert space all integrated squared (relative to $F(d\lambda)\times F_1(d\mu)$) functions, then it it occurs also for certain functions $\psi_0(\lambda,\mu)$ - the projection of cell/element $\psi(\lambda,\mu)$ on subspace $L_{T\times T}(F\times F_1)$.

Let us consider now equivalent measures P and P_1. Let us select everywhere dense in set T the sequence of points t_1, t_2 .... It is obvious, value $\eta_k=\xi(t_k)$, k = 1, 2, ..., form full/total/complete system in hilbert space $L_T(F)$.

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Let us consider densities $p_n(\omega)=P_1(d\omega)/P(d\omega)$ on in σ-algebra $\mathfrak{A}_n$, each of which is generated by values η_k, $k=1,\ldots,n$. By formula (2.1)

$$\log p_n(\omega)=\mathsf{M}\log p_n-\frac{1}{2}\sum_{k,j=1}c_{kj}[\xi(t_k)\xi(t_j)-B(t_k,t_j)], \tag{3.11}$$

where $\{c_{kj}\}$ there is a difference in the matrix/dies, reverse to correlation matrix/dies $\{B_1(t_k,t_j)\}$ and $\{B(t_k,t_j)\}$. The corresponding values

$$\eta_n(\omega)=\sum_{k,j=1}^{n}c_{kj}[\xi(t_k)\xi(t_j)-B(t_k,t_j)],$$

figuring in formula (3.11), belong to space H_2 (T) and represented in the form (3.6):

$$\eta_n=\int\int\psi_n(\lambda,\mu)\Psi(d\lambda,d\mu), \tag{3.12}$$

where

$$\psi_n(\lambda,\mu)=\sum_{k,j=1}^{n}c_{kj}e^{i(\lambda t_k-\mu t_j)}. \tag{3.13}$$

It is easy to check that each of the functions $\psi_n(\lambda, \mu)$ satisfies equation of type (3.9):

$$\int\int e^{-i(\lambda s-\mu t)}\psi_n(\lambda, \mu)\,F(d\lambda)\,F_1(d\mu)=b(s, t) \quad (3.14)$$

with $s, t=t_1, \ldots, t_n$. In fact, this equality can be rewritten in the matrix form:

$$\{B(t_k, t_j)\}\{c_{kj}\}\{B_1(t_k, t_j)\}=\{b(t_k, t_j)\},$$

where

$$\{c_{kj}\}=\{B_1(t_k, t_j)\}^{-1}-\{B(t_k, t_j)\}^{-1},$$

and immediately evident that

$$\{B(t_k, t_j)\}\{c_{kj}\}=\{B(t_k, t_j)\}\{B_1(t_k, t_j)\}^{-1}-E,$$

$$\{B(t_k, t_j)\}\{c_{kj}\}\{B_1(t_k, t_j)\}=$$
$$=\{B(t_k, t_j)\}-\{B_1(t_k, t_j)\}=\{b(t_k, t_j)\}.$$

Equality (3.14) can be rewritten, also, in the form

$$\langle\varphi, \psi_n\rangle_{F\times F_1}=b(s, t), \quad s, t\in T_n,$$

where $T_n=\{t_1, \ldots, t_n\}$, a $\varphi(\lambda, \mu)=e^{i(\lambda s-\mu t)}$, $s, t\in T_n$.

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It is clear that with $m \leqslant n$ the function $\psi_m(\lambda, \mu)$ coincides with the projection of cell/element $\psi_n(\lambda, \mu) \in L_{T_n \times T_n}(F \times F_1)$ on subspace $L_{T_m \times T_m}(F \times F_1)$, so that

$$\| \psi_n - \psi_m \|^2_{F \times F_1} = \| \psi_m \|^2_{F \times F_1} - \| \psi_n \|^2_{F \times F_1} \to 0$$

with $m, n \to \infty$, since sequence $\| \psi_n \|^2_{F \times F_1}$, n = 1, 2, ..., turns out to be monotonically decreasing (and bounded below by zero) and there exists limit $\lim \| \psi_n \|^2$. It is clear also, that since the hilbert space $L_{T \times T}(F \times F_1)$ coincides with the closing/shorting of the being widened spaces $L_{T_n \times T_n}(F \times F_1)$, $n = 1, 2, \ldots$ whereupon each of the functions $\psi_n(\lambda, \mu)$ in relationship/ratio (3.14) it is the projection of cell/element $\psi(\lambda, \mu) \in L_{T \times T}(F \times F_1)$ from equation (3.9), and limit $\lim_{n \to \infty} \psi_n(\lambda, \mu) \in L_{T \times T}(F \times F_1)$ possesses those property that

$$\psi(\lambda, \mu) = \lim_{n \to \infty} \psi_n(\lambda, \mu).$$

This indicates that values η_n form (3.12) figuring in formula (3.11), converge in mean to value

$$\eta = \int \int \psi(\lambda, \mu) \Psi(d\lambda, d\mu) \in H_2(T).$$

Earlier it was shown (see (1.33)), that for equivalent measures there is the limit

$$\lim_{n \to \infty} \mathbf{M} \log p_n.$$

Thus, the unknown density $p(\omega)=P_1(d\omega)/P(d\omega)$ **in** σ**-algebra** $\mathfrak{A}(T)$ **can be determined from asymptotic relation (1.34):**

$$\begin{aligned}\log p(\omega) &= \lim_{n\to\infty} \mathrm{M}\log p_n + \lim_{n\to\infty}[\log p_n - \mathrm{M}\log p_n] = \\ &= \lim_{n\to\infty}\mathrm{M}\log p_n - \frac{1}{2}\lim_{n\to\infty}\eta_n(\omega) = \\ &= \lim_{n\to\infty}\mathrm{M}\log p_n - \frac{1}{2}\lim_{n\to\infty}\int\int\psi_n(\lambda,\mu)\,\Psi(d\lambda,d\mu) = \\ &= \log D - \frac{1}{2}\int\int\psi(\lambda,\mu)\,\Psi(d\lambda,d\mu),\end{aligned}$$

that also it gives to us formula (3.10).

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§4. Further studies of equivalency conditions

1. Gaussian measures which differ by the mean value.

Let us turn to equivalency conditions of Gaussian measures **P** and **P_1** in σ-algebra $\mathfrak{A}$(T) established in theorum 7. Let us examine the case where the spectral measure F(dλ)is absolutely continuous and has a limited density

$$f(\lambda)=\frac{F(d\lambda)}{d\lambda}.$$

For the equivalency of Gaussian measures **P** and **P_1** which differ only by the mean value

$$a(t)=\mathrm{M}_1\xi(t),\quad t\in T, \tag{4.1}$$

it is necessary and sufficient for the function a(t) to permit presentation of the form

$$a(t)=\int e^{-i\lambda t}\psi(\lambda)f(\lambda)\,d\lambda,\quad t\in T, \tag{4.2}$$

where $\psi(\lambda)$ - some function which satisfies the condition

$$\int|\psi(\lambda)|^2 f(\lambda)\,d\lambda<\infty$$

(see theorum 7).

We assume

$$\varphi(\lambda) = \psi(\lambda)\cdot f(\lambda). \tag{4.3}$$

The right side in formula (4.2) determines some function

$$a(t) = \int e^{-i\lambda t}\varphi(\lambda)\,d\lambda, \quad -\infty < t < \infty, \tag{4.4}$$

which coincides with $t \in T$ with the mean value (4.1), in which regard function $\phi(\lambda)$ of the form (4.3) can be integrated in a square

$$\int |\varphi(\lambda)|^2\,d\lambda \leqslant C\int |\psi(\lambda)|^2 f(\lambda)\,d\lambda < \infty$$

with $f(\lambda) \leq C$.

This shows that the function determined by formula (4.4) a(t), $-\infty<t<\infty$, is integratable in a square, and $\phi(\lambda)$ is its Fourier transform:

$$\varphi(\lambda) = \frac{1}{2\pi}\int e^{i\lambda t}a(t)\,dt. \tag{4.5}$$

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As is evident from formula (4.3), the function $\phi(\lambda)$ itself is not only integratable in a square but it also satisfies the condition

$$\int \frac{|\varphi(\lambda)|^2}{f(\lambda)}\,d\lambda < \infty. \tag{4.6}$$

Next, let us assume that the function a(t), $t \in T$, can be extended to the entire real straight line $-\infty<t<\infty$ in such a way that its extension will be integratable in a square by the function a(t), $-\infty<t<\infty$ whose Fourier transform satisfies condition (4.6). Then, with all t

$$a(t) = \int e^{-i\lambda t}\varphi(\lambda)\,d\lambda = \int e^{-i\lambda t}\psi(\lambda)\,f(\lambda)\,d\lambda,$$

where the function $\psi(\lambda) = \varphi(\lambda)/f(\lambda)$ satisfies the condition $\int |\psi(\lambda)|^2 f(\lambda)\,d\lambda < \infty$. From theorum 7 this signifies the equivalency of Gaussian measures P and P_1.

Thus, the following result has been established:

Theorum 9. For the equivalency of Gaussian measures P and P_1 it is necessary and sufficient that function a(t), $t \in T$ (mean value) can be extended to the function a(t), $-\infty<t<\infty$ which is integratable in a square, whose Fourier transform would satisfy condition (4.6).

Next, we will consider that set T is a final segment on a real straight line (let us say, $T=[0, \tau]$).

Let us consider the spectral densities $f(\lambda)$ which satisfy the condition

$$f(\lambda) \leqslant K(1+\lambda^2)^{-n} \quad (4.7)$$

or the condition

$$f(\lambda) \geqslant k(1+\lambda^2)^{-n}, \quad (4.8)$$

where K and k - some positive constants (see

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p. 3 of this paragraph) are common for the case, when spectral densities they satisfy only the condition

$$\overline{\lim_{\lambda\to\infty}} f(\lambda)|\lambda|^{2n} < \infty \quad (4.9)$$

or the condition

$$\underline{\lim_{\lambda\to\infty}} f(\lambda)|\lambda|^{2n} > 0. \quad (4.10)$$

Theorem 10. At spectral density $f(\lambda)$ of type (4.7) for the equivalency of Gaussian measures P and P_1 it is necessary that the function $a(t)$ (average value) would have on the segment $T=[0, \tau]$

question) absolutely continuous (n - 1)-th derived $a^{(n-1)}(t)$ such, that

$$\int_T [a^{(n)}(t)]^2\, dt < \infty. \qquad (4.11)$$

At spectral density f (λ) of type (4.8) the condition indicated is sufficient for a equivalency P and P_1.

Proof. It is obvious, under condition (4.7) the satisfying relationship/ratio (4.6) function ϕ (λ) is such, that

$$\int (1+|\lambda|^n)^2 |\varphi(\lambda)|^2\, d\lambda < \infty, \qquad (4.12)$$

and with any m = 0, 1, ..., n - 1

$$\int |\lambda|^m |\varphi(\lambda)|\, d\lambda \leqslant \leqslant \left[\int \left(\frac{|\lambda|^m}{1+|\lambda|^n}\right)^2 d\lambda\right]^{1/2} \left[\int (1+|\lambda|^n)^2 |\varphi(\lambda)|^2\, d\lambda\right]^{1/2} < \infty.$$

consequently, the function

$$a(t) = \int e^{-i\lambda t} \varphi(\lambda)\, d\lambda$$

has n - 1 derivatives. Let us show that (n - 1) ~~-4~~ th derivative

$$a^{(n-1)}(t) = \int (-i\lambda)^{n-1} e^{-i\lambda t} \varphi(\lambda)\, d\lambda$$

is absolutely continuous, whereupon is satisfied condition (4.11).

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For any nonintersecting intervals (t_k, t_{k+1}), k = 1, 2, ..., we have

$$\sum_k [a^{(n-1)}(t_{k+1}) - a^{(n-1)}(t_k)] = \\ = \int \left[\sum_k (e^{-i\lambda t_{k+1}} - e^{-i\lambda t_k})\right] (-i\lambda)^{n-1} \varphi(\lambda)\, d\lambda = \\ = \int \left[\sum_k \int_{t_k}^{t_{k+1}} (-i\lambda) e^{-i\lambda t}\, dt\right] (-i\lambda)^{n-1} \varphi(\lambda)\, d\lambda = \\ = \int \varphi_\Delta(\lambda)(-i\lambda)^n \varphi(\lambda)\, d\lambda, \qquad (4.13)$$

where

$$\varphi_\Delta(\lambda) = \int e^{-i\lambda t} \chi_\Delta(t)\, dt$$

there is a Fourier transform function $\chi_\Delta(t)$ - the indicator of set $\Delta = \bigcup_k (t_k, t_{k+1})$.

Is obvious, on the strength of condition (4.12)

$$\left|\sum_k [a^{(n-1)}(t_{k+1}) - a^{(n-1)}(t_k)]\right| \leqslant \\ \leqslant \left[\int |\chi_\Delta(\lambda)|^2 d\lambda\right]^{1/2} \left[\int |\lambda|^{2n} |\varphi(\lambda)|^2 d\lambda\right]^{1/2} \leqslant \\ \leqslant C \left[\int |\chi_\Delta(t)|^2 dt\right]^{1/2} = C\delta^{1/2},$$

where C - certain constant, and

$$\delta = \sum_k (t_{k+1} - t_k),$$

that also it proves the absolute continuity of function $a^{(n-1)}(t)$. It is obvious also that its derivative

$$a^{(n)}(t) = \int (-i\lambda)^n e^{-i\lambda t} \varphi(\lambda)\, d\lambda$$

(determined with almost all t) coincides with Fourier transform the integrated squared function $(-i\lambda)^n \varphi(\lambda)$ and therefore satisfies condition (4.11).

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Let now spectral density f (λ) satisfy relationship/ratio (4.8) and is carried out condition (4.11). It is obvious, function a (t) it can be continued from segment T = [0, τ] to entire real straight line in such a way that it would have (n - 1)-th absolutely continuous derivative $a^{(n-1)}(t)$ and would be converted in 0 outside certain segment $T' \supseteq T$, whereupon $\int [a^{(n)}(t)]^2 dt < \infty$. Let us assume

$$\psi(\lambda) = \frac{1}{2\pi} \int e^{i\lambda t} a^{(n)}(t)\, dt.$$

After n-fold integration by parts, we have

$$\varphi(\lambda) = \frac{1}{2\pi} \int e^{i\lambda t} a(t)\, dt = (i\lambda)^{-n} \psi(\lambda)$$

and

$$\int |\lambda|^{2n} |\varphi(\lambda)|^2 d\lambda = \int |\psi(\lambda)|^2 d\lambda = \frac{1}{2\pi} \int [a^{(n)}(t)]^2 dt < \infty.$$

Consequently, is satisfied condition (4.6) and, according to theorem 9, Gaussian measures P and P_1 will be equivalent, Q. E. D.

2. Gaussian measures, which differ in terms of correlation functions. Let us turn to the conditions of equivalency of Gaussian measures P and P_1 on algebra $\mathfrak{A}(T)$, establish/installed in theorem 8. Let us consider the case, when the spectral measures F (dλ) and F_1 (dλ) are absolutely continuous and have the limited densities

$$f(\lambda) = F(d\lambda)/d\lambda \text{ and } f_1(\lambda) = F_1(d\lambda)/d\lambda.$$

Let

$$b(s, t) = B(s, t) - B_1(s, t), \qquad s, t \in T,$$

be a difference in the corresponding correlation functions

$$B(s, t) = \int e^{i\lambda(s-t)} f(\lambda)\, d\lambda$$

and

$$B_1(s, t) = \int e^{i\lambda(s-t)} f_1(\lambda)\, d\lambda.$$

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Theorem 11. For the equivalency of Gaussian measures P and P_1 it is necessary and sufficiently in order that a difference in their

correlation functions b (s, t), s, t $\in$ T, could be continued into the integrated squared function b (s, t) (on an entire plane - < s, t $<$ -), Fourier transform by which

$$\varphi(\lambda, \mu)=\frac{1}{4\pi^2}\int\int e^{i(\lambda s-\mu t)}b(s, t)\,ds\,dt$$

would satisfy the condition

$$\int\int\frac{|\varphi(\lambda, \mu)|^2}{f(\lambda)f_1(\mu)}\,d\lambda\,d\mu<\infty. \quad (4.14)$$

Proof. Accoriing to theorem 8 for a equivalency P and P_1 it is necessary and sufficiently in order that function b (s, t), s, t $\in$ T, would allow/assume representation of the form

$$b(s, t)=\int e^{-i(\lambda s-\mu t)}\psi(\lambda, \mu)f(\lambda)f_1(\mu)\,d\lambda\,d\mu,$$

where $\psi(\lambda, \mu)$ - certain function, satisfying the condition

$$\int\int|\psi(\lambda, \mu)|^2 f(\lambda)f_1(\mu)\,d\lambda\,d\mu<\infty.$$

Let us assume

$$\varphi(\lambda, \mu)=\psi(\lambda, \mu)f(\lambda)f_1(\mu).$$

In the case of the limited densities f (λ) and $f_1(\mu)$ the function which satisfies condition (4.14), turns out to be that which is integrated squarei, but its Fourier transform b (s, t) coincides on set T X T with a difference in the correlation functions in question, so that the conditions of theorem 11 are necessary. On the other hand, if these conditions are satisfied, then is function b (s, t), s, t $\in$ T, represented in the form

$$b(s,t)=\int e^{-i(\lambda s-\mu t)}\varphi(\lambda,\mu)\,d\lambda\,d\mu$$

(where $\varphi(\lambda,\mu)$ satisfies condition (4.14)), and this as it was already said above, indicates the equivalency of Gaussian measures P and P_1. Theorem is demonstrated.

Let us note that in the case, when set T is entire/all real straight line, function b (s,t) is already assigned on an entire plane $-\infty < s, t < \infty$ by the formula

$$b(s,t)=\int\int e^{-i\lambda(s-t)}[f(\lambda)-f_1(\lambda)]\,d\lambda. \qquad (4.15)$$

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It is obvious, if the difference

$$g(\lambda)=f(\lambda)-f_1(\lambda), \qquad -\infty<\lambda<\infty,$$

is different from 0 on the set of positive measure, then

$$\int\int|b(s,t)|^2\,ds\,dt=\infty,$$

and, according to theorem 11, measures P and P_1 will be orthogonal. Consequently, Gaussian measures P and P_1 are equivalent in σ-algebra $\mathfrak{A}(T)$ with $T = (-\infty, \infty)$ if and only if they

coincide (when coincide spectral densities f (λ) and f_1 (λ)), which will be coordinated with an example on page 119.

Subsequently let us examine only the case, when T is the final segment on real lines (let us say, T = [0, τ]), and spectral density f (λ) with certain n satisfies relationship/ratio (4.8). The obtained below results will be common for the case of spectral density f (λ) of type (4.10) in following p. 3.

Let us note that if spectral density f (λ) decreases faster than any degree $|\lambda|^{-2n}$ (with λ → ∞), more precise, if the stationary process ξ (t) is "analytical":

$$\mathsf{M}\left[\xi(t)-\sum_{k=1}^{n}\frac{\xi^{(k)}(0)}{k!}t^{k}\right]^{2}=\int\left|e^{i\lambda t}-\sum_{k=1}^{n}\frac{(i\lambda)^{k}}{k!}t^{k}\right|^{2}f(\lambda)\,d\lambda\leqslant$$
$$\leqslant\left(\frac{|t|^{n+1}}{(n+1)!}\right)^{2}\int\lambda^{2(n+1)}f(\lambda)\,d\lambda\to 0 \text{ with } n\to\infty, \quad (4.16)$$

that addition/completion of σ-algebra $\mathfrak{A}(T)$ contains entire σ-algebra $\mathfrak{A}(-\infty, \infty)$, and, as soon as which was shown above, the equivalency of measures P and P_1 occurs when and only when P and P_1 simply they coincide.

As it was shown earlier (see Section 1 §1 of chapter II), under

condition (4.8) the space $L_T(F)$ consists of the whole analytic functions.

Let us return to research of the conditions of equivalency, data by theorem 4, and let us turn to the appropriate operator $\Delta = E - A^*A$ in hilbert space $L_T(F)$.

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Lemma 5. Let space $L_T(F)$ consist of the whole analytic functions, and let $\Delta = E - A^*A$ there is an operator of Gil'berta - Schmidt. Then is satisfied condition (1.27), equivalent to the fact that the operator Δ does not have equal to 1 eigenvalue.

Proof. On the strength of operator's limitedness $A^*A = E - \Delta$ we have

$$\langle A^*A\varphi, \varphi\rangle_F = \|\varphi\|^2_{F_1} \leqslant C\|\varphi\|^2_F$$

for all $\varphi(\lambda) \in L_T(F)$ (C - certain constant). Let us assume that there is eigenfunction $\varphi(\lambda) \in L_T(F)$ with equal to 1 eigenvalue. Let $\varphi_n(\lambda)$, n = 1, 2, ..., certain sequence of functions from L^0_T, that converges to $\varphi(\lambda)$ in space $L_T(F_1)$. This sequence in $L_T(F)$ converge to 0:

$$\|\varphi_n\|^2_{F_1} = \langle\varphi_n, \varphi_n\rangle_F - \langle[E - A^*A]\varphi_n, \varphi_n\rangle_F \to 0$$

with n $\to \infty$. Consequently, limit function $\varphi(\lambda)$ is equal to 0 with almost all λ, for which $f_1(\lambda) > 0$. But then analytic function $\varphi(\lambda)$

is identically equal to 0, and this contradicts the fact that it there is eigenfunction. Lemma is demonstrated.

This result makes it possible to amplify theorems 4, 5, after drop/omitting in them further condition (1.27) (more precise, this condition is satisfied automatically, if only $\Delta = E - A^* A$ is an operator of Gil'berta - Schmidt).

In turn, this makes it possible to obtain the following result (almost repeating theorem 8):

for the equivalency of Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}(T)$ necessary and sufficiently, in order to difference b (s, t) = B (s, t) − B_1 (s, t) correlation functions would allow/assume representation of the form (comp. 3.9))

$$b(s, t) = \int\int e^{-i(\lambda s - \mu t)} \varphi(\lambda, \mu) F(d\lambda) F(d\mu) \quad (4.17)$$

with s, t $\in$ T, where the function $\varphi(\lambda, \mu)$ is such, that

$$\int\int |\varphi(\lambda, \mu)|^2 F(d\lambda) F(d\mu) < \infty. \quad (4.18)$$

Hence easily is derive/concluded (exactly as theorem 11) the following result.

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Theorem 12. Under condition (4.8) for the equivalency of Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}(T)$ it is necessary and sufficiently, in order to a difference in their correlation functions b (s, t), s, t $\in$ T (where T = [0, τ]), I could be continued into the integrated squared function b (s, t), $-\infty$ < s, t < + ∞, Fourier transform by which would satisfy condition [1])

$$\int\int \frac{|\varphi(\lambda, \mu)|^2}{f(\lambda) f(\mu)} d\lambda\, d\mu < \infty. \tag{4.19}$$

FOOTNOTE [1]. It is clear that theorem 12 remains valid, when space $L_T(F)$ consists of analytic functions (see Chapter II). ENDFOOTNOTE.

Let us turn now to condition (4.19). Let us consider the function

$$\varphi(\lambda, \mu) = \frac{1}{4\pi^2}\int\int e^{i(\lambda s - \mu t)} b(s, t)\, ds\, dt, \tag{4.20}$$

figuring under this condition (where b (s, t) is the corresponding continuation assigned on square T X T of a difference in the correlation functions).

Let spectral density f (λ) satisfy relationship/ratio (4.7). Then from condition (4.19) it follows that

$$\int\int (1+|\lambda|^n)^2(1+|\mu|^n)^2|\varphi(\lambda,\mu)|^2\,d\lambda\,d\mu<\infty \quad (4.21)$$

and with any k, m = 0, 1, ..., n - 1

$$\int\int|\lambda|^k|\mu|^m|\varphi(\lambda,\mu)|\,d\lambda\,d\mu\leqslant$$
$$\leqslant\left[\int\left(\frac{|\lambda|^k}{1+|\lambda|^n}\right)^2 d\lambda\int\left(\frac{|\mu|^m}{1+|\mu|^n}\right)^2 d\mu\right]^{1/2}\times$$
$$\times\left[\int\int(1+|\lambda|^n)^2(1+|\mu|^n)^2|\varphi(\lambda,\mu)|^2\,d\lambda\,d\mu\right]^{1/2}<\infty.$$

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Consequently, the function

$$b(s,t)=\int\int e^{-i(\lambda s-\mu t)}\varphi(\lambda,\mu)\,d\lambda\,d\mu$$

has all derivatives to order n - 1 in terms of each variable:

$$\frac{\partial^{k+m}b(s,t)}{\partial s^k\,\partial t^m}=\int\int(-i\lambda)^k(i\mu)^m e^{-i(\lambda s-\mu t)}\varphi(\lambda,\mu)\,ds\,dt$$
$$(k,\ m=0,\ \ldots,\ n-1).$$

Let us show that the function $C(s,t)=\frac{\partial^{2(n-1)}b(s,t)}{\partial s^{n-1}\,\partial t^{n-1}}$ is absolutely continuous. This means that C (s,t) is the "function of distribution" of certain Borel absolutely continuous measure [1] m (Δ) on square T X

T, i.e.,

$$m(\Delta)=C(s'',t'')-C(s'',t')-C(s',t'')+C(s',t')$$

for any rectangle Δ = (s', s"] X (t' t"].

FOOTNOTE [1]. In other words, the additive function of bounded variation m (Δ) it is absolutely continuous relative to Lebegovskaya measure. ENDFOOTNOTE.

It is easy to see that for any multitude $\Delta=\bigcup_k\Delta_k$, which is the association of the finite number of nonintersecting rectangles $\Delta_k=(s_k',s_k'']\times(t_k',t_k'']$, occurs the following equality:

$$m(\Delta)=\int\int\varphi_\Delta(\lambda,\mu)(-i\lambda)^n(i\mu)^n\varphi(\lambda,\mu)\,d\lambda\,d\mu,$$

where

$$\varphi_\Delta(\lambda,\mu)=\int\int e^{-i(\lambda s-\mu t)}\chi_\Delta(s,t)\,ds\,dt$$

there is a Fourier transform function $\chi_\Delta(s,t)$ - the indicator of set Δ (comp. (4.13)). On the strength of condition (4.21)

$$|m(\Delta)|\leqslant$$
$$\leqslant\left[\int\int|\varphi_\Delta(\lambda,\mu)|^2d\lambda\,d\mu\right]^{1/2}\left[\int\int|\lambda|^{2n}|\mu|^{2n}|\varphi(\lambda,\mu)|^2d\lambda\,d\mu\right]^{1/2}\leqslant$$
$$\leqslant C\left[\int\int|\chi_\Delta(s,t)|^2ds\,dt\right]^{1/2}Cl(\Delta)^{1/2},$$

where l (Δ) - is Lebegovskaya measure of a set Δ. It is evident that additive function m (Δ) is really/actually absolutely continuous relative to Lebegovskaya measure.

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It is clear that density c (s, t) = m (ds dt) /ds dt coincides with derivative $\frac{\partial^{2n} b(s, t)}{\partial s^n \partial t^n}$ (by existing with almost all s, t $\in$ T X T). It is obvious, under condition (4.21) this derivative $c(s, t) = \frac{\partial^{2n} b(s, t)}{\partial s^n \partial t^n}$ is Fourier transform the integrated squared function $\psi(\lambda, \mu) = (-i\lambda)^n (i\mu)^n \varphi(\lambda, \mu)$, namely:

$$\frac{\partial^{2n} b(s, t)}{\partial s^n \partial t^n} = \int\int (-i\lambda)^n (i\mu)^n e^{-i(\lambda s - \mu t)} \varphi(\lambda, \mu) \, d\lambda \, d\mu,$$

and, therefore,

$$\iint\limits_{T \times T} \left[\frac{\partial^{2n} b(s, t)}{\partial s^n \partial t^n} \right]^2 ds \, dt < \infty. \qquad (4.22)$$

Let now spectral density f (λ) satisfy relationship/ratio (4.8), and a difference in the correlation functions b (s, t) has absolutely continuous derivative $\frac{\partial^{2(n-1)} b(s, t)}{\partial s^{n-1} \partial t^{n-1}}$, whereupon is satisfied

condition (4.22).

It is obvious, function b (s, t) can be continued to entire plane - < s, t < - in such a way that it would have as before the type indicated derivative and would be converted in 0 outside certain final square T' X T' $\supseteq$ T X T. Using a repeated integration and and 2n-multiple integration in parts, from the formula

$$\psi(\lambda, \mu) = \frac{1}{4\pi^2} \iint\limits_{T' \times T'} e^{i(\lambda s - \mu t)} \frac{\partial^{2n}}{\partial s^n \partial t^n} b(s, t)\, ds\, dt$$

we obtain, that

$$\varphi(\lambda, \mu) = \frac{1}{4\pi^2} \iint\limits_{T' \times T'} e^{i(\lambda s - \mu t)} b(s, t)\, ds\, dt = \frac{1}{(i\lambda)^n (-i\mu)^n} \psi(\lambda, \mu).$$

Consequently,

$$\int\limits_{-\infty}^{\infty}\!\!\int |\lambda|^{2n} |\mu|^{2n} |\varphi(\lambda, \mu)|^2 d\lambda\, d\mu = \int\limits_{-\infty}^{\infty}\!\!\int |\psi(\lambda, \mu)|^2 d\lambda\, d\mu < \infty,$$

whence for spectral density f (λ) of form (4.8) escape/ensues condition (4.19) of theorem 12.

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Thus, for density f (λ) of type (4.7) from condition (4.19) escape/ensues condition (4.22), and for a density of type (4.8), on the contrary, from condition (4.22) escape/ensues condition (4.19), which means that the operator Δ in hilbert space $L_T(F)$, determined

by the appropriate function

$$b(s, t) = \langle \Delta e^{i\lambda s}, e^{i\lambda t} \rangle_F, \quad s, t \in T,$$

is the operator of Gil'berta - Schmidt.

Recall now that examine/considered by us function b (s, t) depends only on difference alternating/variable s are t (see formula (4.15)).

It is clear that the absolute continuity of function $\frac{\partial^{2(n-1)} b(s, t)}{\partial s^{n-1} \partial t^{n-1}}$ (of the pair of the variables s, t) is equivalent to the presence 2n - by 1 absolutely continuous derivative $b^{(2n-1)}(t), \ -\tau \leqslant t \leqslant \tau,$ of the difference

$$b(t) = B(t) - B_1(t), \quad -\tau < t < \tau,$$

the correlation functions B (t) [redacted] and B_1 (t), condition (4.22) meaning that

$$\iint\limits_{T \times T} [b^{(2n)}(s-t)]^2 \, ds \, dt < \infty. \qquad (4.23)$$

In summation, we obtained the following result.

Theorem 13. For the equivalency of Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}(T)$ in the case of spectral density f (λ) of type (4.7) it is necessary that the difference in the correlation functions b

(t) = B (t) - B_1 (t) would have in interval $(-\tau, \tau)$ derivatives to order 2n - 1, whereupon (2n - 1) -4 derivative $b^{(2n-1)}(t)$ would be absolutely continuous, and its (existing with almost all t) derivative $b^{(2n)}(t)$ satisfied condition (4.23). In the case of spectral density f (λ) of type (4.8) these conditions are sufficient for equivalency.

Let us note that, agreement, in the case of a spectral density of the type

$$f(\lambda) \asymp (1+\lambda^2)^{-n} \quad (4.24)$$

condition (4.23) is necessary and sufficient for the equivalency of Gaussian measures P and P_1)[1].

FOOTNOTE [1]. In connection with condition (4.23) comp. the work of Feldman (G Feldman, Some slasses of equivalent Gaussian processes on an interval, Pacif. J Math., 10 (1960) 1200-1220). ENDFOOTNOTE.

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Example. Let $f(\lambda) = \frac{\sigma^2}{\pi}\frac{\alpha}{(\lambda^2+\alpha^2)}$, where σ^2 and some α- positive parameters. The corresponding correlation function is

$$B(t) = \sigma^2 e^{-\alpha|t|}.$$

Let the correlation function B_1 (t) take the form

$$B_1(t)=\begin{cases}1-|t| & \text{with } |t|\leqslant 1,\\ 0 & |t|>1\end{cases}$$

(see Fig. 1) corresponding spectral density is

$$f_1(\lambda)=\frac{1}{\pi}\,\frac{1-\cos\lambda}{\lambda^2}.$$

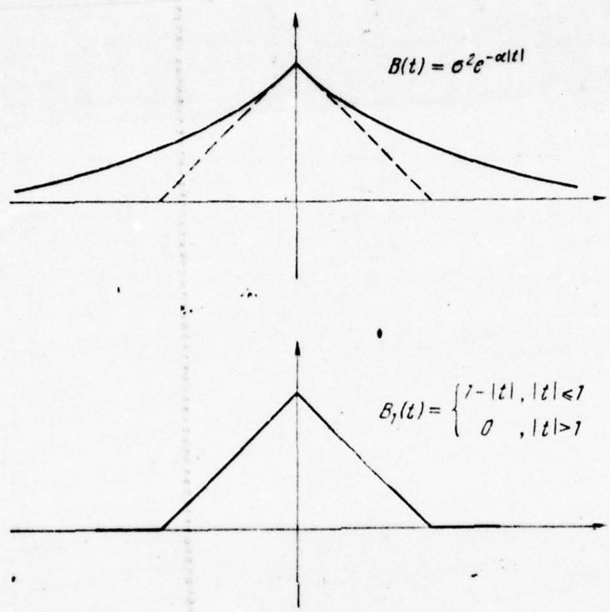

Рис. 1.

Fig. 1.

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It is obvious, condition (4.23) is satisfied (for n = 1), if the parameters σ^2, α and τ are such, that $\sigma^2\alpha = 1$, $\tau \leqslant 1$. Under this condition Gaussian measures P and P_1 are equivalent in σ-algebra

$\mathfrak{A}(T)$, $T=[0,\tau]$. At other values σ^2, α and τ these measures will be orthogonal.

The nonintersecting carriers of orthogonal measures P and P_1 are described in p. 2 §1.

3. Some spectral conditions of equivalency. Obtained above condition (4.23) shows that the equivalency of Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}(T)$ (where T - any final segment) depends only on the behavior of spectral densities "at infinity", it is more precise, for spectral density f (λ) of type (4.8) [redacted] occurs the following simple fact.

Theorem 14. The arbitrary change f (λ) in any finite interval (such, that is obtained spectral density f_1 (λ)) it leads to Gaussian measure P_1 equivalent initial measure P.

Proof. In fact, in this case the difference in the correlation functions is Fourier transform the finite function $g(\lambda) = f(\lambda) - f_1(\lambda)$:

$$b(t)=\int e^{i\lambda t}\, g(\lambda)\, d\lambda, \tag{4.25}$$

it has derived all orders so that, in particular, for any final segment T is satisfied the condition of equivalency (4.23).

This makes it possible to spread the obtained previously results for the case, when spectral density f (λ) satisfies more weakly than (4.8), condition (4.10):

$$\varliminf_{\lambda\to\infty} f(\lambda)\,|\lambda|^{2n} > 0.$$

For example, theorem 9 can be intensified as follows.

Theorem 15. At spectral density f (λ) of type (4.10) for the equivalency Gaussian measures P and P_1 differing in terms of the average value of a (t), t ∈ T (where T is certain segment), it is necessary and sufficient in order that function a (t), t ∈ T, could be continued into the integrated squared function a (t), -∞ < t < ∞, Fourier transform by which with any R < ∞ satisfies the condition

$$\int\limits_{|\lambda|>R} \frac{|\varphi(\lambda)|^2}{f(\lambda)}\,d\lambda < \infty. \tag{4.26}$$

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Proof. Let $\tilde{f}$ (λ) satisfy condition (4.8) and coincides with f (λ) with $|\lambda| > R$. Let $\tilde{P}$ there is a Gaussian measure with zero average value and spectral density $\tilde{f}$ (λ). As it was shown above, measures P and $\tilde{P}$ were equivalent, so that equivalency P and P was equivalent equivalency P_1 and $\tilde{P}$. But according to theorem 9 for a

equivalency P_1 and $\tilde{P}$ it is necessary and sufficiently condition (4.6):

$$\int_{-\infty}^{\infty} \frac{|\varphi(\lambda)|^2}{f(\lambda)} d\lambda < \infty,$$

which on the strength of the relationship/ratio

$$f(\lambda) \asymp 1 \quad \text{with} \, |\lambda| \leqslant R$$

equivalent to the condition

$$\int_{|\lambda|>R} \frac{|\varphi(\lambda)|^2}{f(\lambda)} d\lambda < \infty,$$

that also it gives to us condition (4.26), since $\tilde{f}(\lambda) = f(\lambda)$ with $|\lambda| > R$. Theorem is demonstrated.

Let us consider spectral densities f (λ), that satisfy weaker than (4.7), to condition (4.9):

$$\overline{\lim_{\lambda\to\infty}} f(\lambda)|\lambda|^{2n} < \infty.$$

Any this density f (λ) coincides with certain density $\hat{f}(\lambda)$ of type (4.7) with sufficiently large $|\lambda|$:

$$f(\lambda) = f(\lambda) \quad \text{with} \, |\lambda| > R.$$

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Using the same method with the introduction of the corresponding Gaussian measure $\tilde{P}$, that and in the proof of theorem 15, it is

possible to as follows generalize theorem 10:

condition (4.11) is necessary for the equivalency of Gaussian measures P and P_1 if spectral density $\hat{f}(\lambda)$ then so that it would satisfy relationship/ratio (4.8) and would coincide with f (λ) with $|\lambda| > R$. It is clear that

condition (4.11) is sufficient for the equivalency of Gaussian measures P and P_1 if spectral density f (λ) satisfies relationship/ratio (4.10).

Let us turn now to Gaussian measures P and P_1 with zero mean values and spectral densities f (λ) and $f_1(\lambda)$, from which f (λ) is limited and belongs to type (4.10).

It is analogous with that, as this was done above, let us consider Gaussian measure $\tilde{P}$ with the limited spectral density $\tilde{f}(\lambda)$ of type (4.8), which coincides with f (λ) outside any interval $|\lambda| \leqslant R$. The Gaussian measures P and $\tilde{P}$ are equivalent, so that the initial measures P_1 and P will be equivalent when and only when this same property will possess measures P_1 and $\tilde{P}$. For the equivalency of Gaussian measures P_1 and $\tilde{P}$ it is necessary and sufficiently condition (4.19):

$$\int_{-\infty}^{\infty}\int \frac{|\tilde{\varphi}(\lambda,\mu)|^2}{\tilde{f}(\lambda)\tilde{f}(\mu)}\, d\lambda\, d\mu < \infty,$$

where $\tilde{\varphi}(\lambda, \mu)$ there is a Fourier transform of certain continuation of the function

$$b(s, t) = \int e^{i\lambda(s-t)} [\tilde{f}(\lambda) - f_1(\lambda)]\, d\lambda, \qquad t \in T.$$

Since $f(\lambda) \asymp 1$ with $\lambda \leqslant R$ and $\tilde{f}(\lambda) = f(\lambda)$ with $|\lambda| > R$, this condition is equivalent to the fact that

$$\int\limits_{|\lambda| > R} \int\limits_{|\mu| > R} \frac{|\tilde{\varphi}(\lambda, \mu)|^2}{f(\lambda) f(\mu)}\, d\lambda\, d\mu < \infty. \qquad (4.27)$$

end section.

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But function b (s, t) with $s, t \in T$ differs from the function

$$b(s, t)=\int e^{i\lambda(s-t)}[f(\lambda)-f_1(\lambda)]\,d\lambda$$

only in terms of the term of the form

$$c(s, t)=\int e^{i\lambda(s-t)}[\tilde{f}(\lambda)-f(\lambda)]\,d\lambda.$$

Since difference $f(\lambda)-\tilde{f}(\lambda)$ is converted in 0 with $|\lambda|>R$, the function c (s, t) is infinitely differentiated and, obviously, allow/assumes this continuation from square $T\times T$, for which the Fourier transform $\psi(\lambda, \mu)$ satisfies the condition

$$\int\int|\lambda|^{2n}|\mu|^{2n}|\psi(\lambda, \mu)|^2\,d\lambda\,d\mu<\infty,$$

where n is the same as in relationship/ratio (4.10). It is clear that if the function $\tilde{\varphi}(\lambda, \mu)$ satisfies condition (4.27), then the function

$$\varphi(\lambda, \mu)=\tilde{\varphi}(\lambda, \mu)+\psi(\lambda, \mu),$$

being the Fourier transform of the proper continuation of the function

$$b(s, t)=\tilde{b}(s, t)+c(s, t), \quad s, t\in T,$$

will satisfy the condition

$$\int\limits_{|\lambda|>R}\int\limits_{|\mu|>R}\frac{|\varphi(\lambda, \mu)|^2}{f(\lambda)f(\mu)}\,d\lambda\,d\mu<\infty. \qquad (4.28)$$

and, on the contrary, if certain continuation of function b (s, t) has Fourier transform $\varphi(\lambda, \mu)$, satisfying condition (4.28), then thereby property will possess function $\tilde{b}$ (s, t), i.e., Gaussian measures P_1 and $\tilde{P}$ will be equivalent.

We obtained the following generalization of theorem 12.

Theorem 16. Under condition (4.10) for the equivalency of Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}(T)$ (T is any final segment) it is necessary and sufficiently in order that a difference in their correlation functions $b(s, t)$, $s, t \in T$, could be continued into the integrated squared function b (s, t), $-\infty < s, t < \infty$, Fourier transform $\varphi(\lambda, \mu)$ by which would satisfy condition (4.28) with any R.

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Hence easily it follows that theorem 13 occurs not only for spectral densities f (λ), that satisfy appropriate conditions (4.7) and (4.8), but also for wider type densities, which satisfy corresponding conditions (4.9) and (4.10). Specifically,

at spectral density f (λ) of the type

$$0 < \underline{\lim}_{\lambda\to\infty} f(\lambda)\,|\,\lambda\,|^{2n} \leqslant \overline{\lim}_{\lambda\to\infty} f(\lambda)\,|\,\lambda\,|^{2n} < \infty \qquad (4.29)$$

condition (4.23) is necessary and sufficient for the equivalency of Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}(T)$.

Any spectral density f (λ) of type (4.29) is such, that

$$f(\lambda) \asymp |\,\varphi(\lambda)\,|^2 \qquad (4.30)$$

with the sufficiently large $|\lambda|$, where $\varphi(\lambda)$ is a Fourier transform of certain integrated squared finite function:

$$\varphi(\lambda) = \int e^{i\lambda t} c(t)\,dt$$

(c (t) = 0 outside certain finite interval). Moreover, we will show below that condition (4.30) satisfies any density f (λ), such, that with certain r > 1

$$0 < \underline{\lim}_{\lambda\to\infty} f(\lambda)\,|\,\lambda\,|^{r} \leqslant \overline{\lim}_{\lambda\to\infty} f(\lambda)\,|\,\lambda\,|^{r} < \infty \qquad (4.31)$$

(recall that f (λ) - the integrated function).

In fact, the product of any functions of type (4.30) again belongs to this type, since

$$f_1(\lambda) f_2(\lambda) \asymp |\,\varphi_1(\lambda)\,\varphi_2(\lambda)\,|^2,$$

where

$$\varphi_1(\lambda)\,\varphi_2(\lambda) = \int e^{i\lambda t}\,[c_1(t) * c_2(t)]\,dt$$

there is a Fourier transform the finite function

$$c_1(t) * c_2(t) = \int c_1(t-s)\,c_2(s)\,ds,$$

by the being fold of the corresponding finite functions $c_1(t)$ and $c_2(t)$.

(t).

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Therefore it suffices to show that type (4.30) includes all the functions f (λ), that satisfy condition (4.31) with 0 < r < 1. Let us assume

$$\varphi(\lambda) = \int_{-\delta}^{\delta} e^{i\lambda t} |t|^{r-1}\,dt = 2\int_0^{\delta} \cos\lambda t \cdot t^{r-1}\,dt.$$

During the replacement λt = u, we obtain, that

$$\varphi(\lambda) = 2\lambda^{-r}\int_0^{\lambda\delta} \cos u \cdot u^{r-1}\,du \sim k\lambda^{-r}$$

(with λ → ∞), since there is the limit

$$k = \lim_{t\to\infty}\int_0^t \cos u \cdot u^{r-1}\,du \quad (0<r<1).$$

It is evident that for spectral density f (λ) of type (4.31) occurs relationship/ratio (4.30).

Theorem 17. At spectral densities of type (4.30) for the equivalency of Gaussian measures P and P_1 in σ-algebra $\mathfrak{A}(T)$ (where

T is any final segment) it is sufficient in order that the function

$$h(\lambda)=\frac{f(\lambda)-f_1(\lambda)}{f(\lambda)}$$

with which [or R < ∞ would satisfy condition [1]

$$\int\limits_{|\lambda|>R}|h(\lambda)|^2d\lambda<\infty. \qquad (4.32)$$

FOOTNOTE [1]. Let us note that if h (λ) → 0 with λ → ∞, then together with spectral density f (λ) to type (4.30) belongs spectral density f_1 (λ),

$$f_1(\lambda)\asymp f(\lambda)$$

with sufficiently large λ. ENDFOOTNOTE.

Proof. Let us consider first the case, when the function

$$h(\lambda)=\frac{f(\lambda)-f_1(\lambda)}{f(\lambda)}$$

is integrated squared and

$$f(\lambda)=|\varphi(\lambda)|^2,$$

where $\varphi(\lambda)$ is a Fourier transform certain finite function c (t).

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Let us find this continuation of a difference in the correlation functions

$$b(s, t) = \int e^{i\lambda(s-t)} h(\lambda) f(\lambda) d\lambda,$$

for which it will be carried out condition (4.19) of theorem 12.

Using the fact that the Fourier transform of product coincides with the fold of the transforms of factors, we obtain, that

$$b(s, t) = \int\int a(u-v) c(s-u) c(t-v) du\, dv,$$

where a (t) there is a Fourier transform function h (λ). In this representation of function $b(s, t),\ s,\ t \in T$. the finite functions c (s - u) and c (t - v) are converted in 0, when alternating/variable u and v lie/rest outside certain final segment T', so that

$$b(s, t) = \int\int_{T' \times T'} a(u-v) c(s-u) c(t-v) du\, dv, \quad s, t \in T.$$

Let us select this continuation a (u, v),-∞ < u, v < ∞, function $a(u-v),\ u,\ v \in T'$, in order that on an entire plane function a (u, v) would be integrated squared, and let us designate $\psi(\lambda, \mu)$ its Fourier transform. With all s, t let us determine function b (s, t),-∞ < s, t < ∞, after assuming

$$b(s, t) = \int\int a(u, v) c(s-u) c(t-v) du\, dv, \quad -\infty < s, t < \infty.$$

It is obvious, with $s, t \in T$ it coincides with a difference in the

correlation functions, and its Fourier transform

$$\varphi(\lambda, \mu) = \psi(\lambda, \mu) | \varphi(\lambda) |^2$$

satisfies condition (4.19). Thus, in the case in question the Gaussian measures P and P_1 are equivalent.

Let us consider now the arbitrary density f (λ) of type (4.30), after assuming temporarily that

$$f_1(\lambda) \geqslant f(\lambda).$$

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It is clear that with sufficiently large λ (let us say, $|\lambda| > R$) spectral density f (λ) is such, that

$$f(\lambda) \asymp \tilde{f}(\lambda) \text{ and } f(\lambda) \geqslant \tilde{f}(\lambda),$$

where $\tilde{f}$ (λ) - examined above type spectral density $\tilde{f}(\lambda) = |\varphi(\lambda)|^2$, and $\varphi(\lambda)$ - Fourier transform certain finite function. Without limiting generality, it is possible to count that f (λ) = f_1 (λ) with $|\lambda| \leqslant R$, since any Gaussian measure with spectral density which differs from f (λ) only by the finite interval, will be equivalent to the initial Gaussian measure P. During this selection f (λ) the satisfying condition (4.32) function h (λ) will be integrated squared, since h (λ) = 0 with $|\lambda| \leqslant R$. Let us assume

$$\tilde{f}_1(\lambda) = \tilde{f}(\lambda) + [f_1(\lambda) - f(\lambda)].$$

It is obvious,

$$\tilde{h}(\lambda) = \frac{\tilde{f}(\lambda) - \tilde{f}_1(\lambda)}{\tilde{f}(\lambda)} = \frac{f(\lambda) - f_1(\lambda)}{\tilde{f}(\lambda)} = h(\lambda)\frac{f(\lambda)}{\tilde{f}(\lambda)} \asymp h(\lambda),$$

so that the function $\tilde{h}(\lambda)$ is integrated squared. As it was proved above, corresponding to spectral densities $\tilde{f}(\lambda)$ and $\tilde{f}_1(\lambda)$ Gaussian measures $\tilde{P}$ and $\tilde{P}_1$ will be equivalent. Therefore for certain continuation of a difference in the correlation functions

$$b(s, t) = \int e^{i\lambda(s-t)}[\tilde{f}(\lambda) - \tilde{f}_1(\lambda)]\, d\lambda =$$
$$= \int e^{i\lambda(s-t)}[f(\lambda) - f_1(\lambda)]\, d\lambda, \quad s, t \in T,$$

the corresponding Fourier transform $\varphi(\lambda, \mu)$ will satisfy condition (4.28):

$$\int\limits_{|\lambda|>R} \int\limits_{|\mu|>R} \frac{|\varphi(\lambda, \mu)|^2}{\tilde{f}(\lambda)\tilde{f}(\mu)}\, d\lambda\, d\mu < \infty,$$

and that more to the condition

$$\int\limits_{|\lambda|>R} \int\limits_{|\mu|>R} \frac{|\varphi(\lambda, \mu)|^2}{f(\lambda) f(\mu)}\, d\lambda\, d\mu < \infty,$$

since $f(\lambda) \geq f(\lambda)$ with $|\lambda| > R$.

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It is evident that for the Gaussian measures P and P_1 is satisfied the condition of equivalency (4.23), that figures in theorem 16.

For the full/total/complete completion of proof we should now be freed from the time/temporary limitation, according to which $f_1(\lambda) \geq f(\lambda)$. This can be done, after considering Gaussian measure P_2 with spectral density $f_2(\lambda) = f(\lambda) + \max.[0, f_1(\lambda) - f(\lambda)]$. Is obvious, $f_2(\lambda) \geq f(\lambda)$, $f_2(\lambda) \geq f_1(\lambda)$ and with certain $R < \infty$ are satisfied conditions of type (4.32):

$$\int\limits_{|\lambda|>R}\left(\frac{f_2(\lambda)-f(\lambda)}{f(\lambda)}\right)^2 d\lambda<\infty, \qquad \int\limits_{|\lambda|>R}\left(\frac{f_2(\lambda)-f_1(\lambda)}{f_1(\lambda)}\right)^2 d\lambda<\infty$$

(since $f(\lambda)\asymp f_1(\lambda)$ with sufficiently large λ). According to already demonstrated Gaussian measures P and P_2, and also Gaussian measures P_1 and P_2 will be equivalent. Consequently, will be equivalent Gaussian measures P and P_1. Theorem is demonstrated.

In conclusion let us note that condition (4.32) is very close in order to be that necessary for the equivalency of Gaussian measures P and P_1. Specifically, as it is actually shown in p. 261 this chapter, at extremely general type spectral densities (4.31)

Gaussian measures P and P_1 will be orthogonal, if the

corresponding function

$$h(\lambda) = \frac{f(\lambda) - f_1(\lambda)}{f(\lambda)}$$

is such, that

$$\lim_{\lambda \to \infty} h(\lambda) \lambda^{1/2} = \infty \qquad (4.33)$$

(this is the corollary of relationship/ratios of the type (7.10) chapter I) [1].

FOOTNOTE [1]. In connection with theorem 17 and condition (4.33) comp. the works of D. S. Apokorin and V. G. Alekseyev (D. S. Apokorii, Gaussovskiyes the measures, which correspond to the generalized stationary processes theoretical probability, and its uses, XII, No 4 (1967), 698-707; V. G. Alekseyev, Ob the conditions of the mutual singularity of the Gaussian measures, which correspond to two random processes. theoretical probabilities and its uses, VII, No 3 (1963), 304-308). ENDFOOTNOTE.

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Chapter IV.

CONDITIONS OF THE REGULARITY OF STATIONARY RANDOM PROCESSES.

§1. Introduction. Some preliminary information.

Let us consiler stationary in narrow sense random process ξ (t) with the continuous or discrete time t. Let us designate as before $\mathfrak{A}(T)$ σ-algebra of the events, generated by the flow of process on set T, i.e., $\mathfrak{A}(T)$ there is minimum σ-algebra, which contains the events of the form

$$\{\xi(t_1)\in E_1, \ldots, \xi(t_s)\in E_s\}, \quad t_1, \ldots, t_s\in T,$$

E_i essence the borel sets of real straight line [1].

FOOTNOTE [1]. We are limited real processes exclusively for simplicity. All the subsequent theorems are accurate for composite processes. It is necessary to only remember that for composite Gaussian processes $M\xi(t)\xi(s)=0$ with all t, s (see [12]). However, during the study only hilbert

space H (-∞, ∞) of no differences in the examination of composite processes appears, and to us will more conveniently examine this space above the field of complex numbers. ENDFOOTNOTE.

The algebras of form $\mathfrak{A}(-\infty, t)$ determine the past of process (to torque/moment t), of the algebra of form $\mathfrak{A}(t, \infty)$ - future (after torque/moment t).

If with any t > 0 σ-algebra $\mathfrak{A}(-\infty, t)$, $\mathfrak{A}(t+\tau, \infty)$ are independent, then with all $A \in \mathfrak{A}(-\infty, t)$, $B \in \mathfrak{A}(t+\tau, \infty)$

$$\mathsf{P}(AB) - \mathsf{P}(A)\mathsf{P}(B) = 0. \qquad (1.1)$$

In the general case the left side of this equality (or value, similar to it) can be taken for the basis of dependence measurement between σ-algebra $\mathfrak{A}(-\infty, t)$ and $\mathfrak{A}(t+\tau, \infty)$.

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The various kinds of the condition of the weakening of the dependence between $\mathfrak{A}(-\infty, t)$, $\mathfrak{A}(t+\tau, \infty)$ with an increase τ - the conditions of regularity - widely are used during propagation to the stationary processes of the theorems, known for the sequences of independent random quantities [1].

FOOTNOTE [1]. See, for example, [14], [22]. ENDFOOTNOTE.

The most commonly used conditions of regularity are the following conditions.

1. Stationary process ξ (t) is called regular, if σ-algebra

$$\mathfrak{A}(-\infty, -\infty) = \mathfrak{A}(-\infty) \overset{\text{def}}{=} \bigcap_t \mathfrak{A}(-\infty, t)$$

is trivial, i.e., it contains only the events of probability 0 or 1.

Let us note that this condition can be registered as follows in the terms of differences $P(AB) - P(A)P(B)$: for all $B \in \mathfrak{A}(-\infty, \infty)$

$$\sup_{A \in \mathfrak{A}(-\infty, t)} |P(AB) - P(A)P(B)| \xrightarrow[t \to -\infty]{} 0. \quad (1.2)$$

Actually, let the process ξ (t) be regular. Let us designate χ_A the indicator of event A, i.e.,

$$\chi_A = \chi_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

Let, further, the random variables η_1, η_2 be determined by equalities $\eta_1 = \chi_A - P\{A\}$, $\eta_2 = \chi_B - P\{B\}$. Then

$$M\eta_1\eta_2 = P(AB) - P(A)P(B).$$

Random variable η_1 measured relative to $\mathfrak{A}(-\infty, t)$, and, therefore,

$$M\eta_1\eta_2 = M\{\eta_1\}M\{\eta_2 \mid \mathfrak{A}(-\infty, t)\} \leqslant \\ \leqslant (M\eta_1^2)^{1/2}\left[M(M(\eta_2 \mid \mathfrak{A}(-\infty, t))^2\right]^{1/2} \leqslant \\ \leqslant M(M(\eta_2 \mid \mathfrak{A}(-\infty, t)))^2 \xrightarrow[t\to-\infty]{} M(M(\eta_2 \mid \mathfrak{A}(-\infty)))^2 = 0.$$

On the contrary, let process ξ (t) is irregular.

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Then σ-algebra $\mathfrak{A}(-\infty)$ is significant and contains at least one event A, for which $0<P\{A\}<1$. With all t $A\in\mathfrak{A}(-\infty, t)$, so that

$$\sup_{B\in\mathfrak{A}(-\infty, t)} \mid P(AB)-P(A)P(B)\mid \geqslant P\{A\}-P^2\{A\} \nrightarrow 0.$$

If we in (1.1) take sup and on $B\in\mathfrak{A}(t+\tau, \infty)$, we will arrive at the following condition.

2. Stationary process ξ (t) satisfies the condition of powerful mixing [1], if

$$\alpha(\tau)=\sup_{A\in\mathfrak{A}(-\infty, t),\ B\in\mathfrak{A}(t+\tau, \infty)} \mid P(AB)-P(A)P(B)\mid \xrightarrow[\tau\to\infty]{} 0. \quad (1.3)$$

FOOTNOTE [1]. The condition of "powerful mixing" obtained wide reputation because of that success, which was achieve/reached during its use in limit theorems for dependent variables (for example, see [14], [22]); for the first time it was used for this purpose, apparently, Rozenblatt (see M Rosenblatt, A central limit theorem and strong mixing condition, Proc. Nat. Acad. Sci. USA 42 (1956)). ENDFOOTNOTE.

Value α (τ) characterizes the "speed of mixing" and is called the coefficient of mixing.

3. Let us call process ξ (t) absolutely regular, if

$$\beta(\tau)=\mathsf{M}\sup_{A\in\mathfrak{A}(t+\tau,\infty)}|\mathsf{P}\{A\mid\mathfrak{A}(-\infty,t)\}-\mathsf{P}\{A\}|\xrightarrow[\tau\to\infty]{}0. \quad (1.4)$$

Value β (τ) is called the coefficient (absolute) of regularity.

It is easy to see that α (τ) $\leq$ β (τ).

Actually, with all $A\in\mathfrak{A}(t+\tau,\infty)$, $B\in\mathfrak{A}(-\infty,t)$ occurs the inequality

$$|\mathsf{P}(AB)-\mathsf{P}(A)\mathsf{P}(B)|=$$
$$=\left|\int_B(\mathsf{P}\{A\mid\mathfrak{A}(-\infty,t)\}-\mathsf{P}\{A\})\,d\mathsf{P}\right|\leqslant\beta(\tau).$$

Thus, absolute regularity assigns on process ξ (t) more severe limitations, than the condition of powerful mixing. After replacing

in (1.4) averaging on entire space of simple events Ω with taking supremum'a, we let us arrive at even more limiting condition.

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4. Process ξ (t) satisfies the condition of evenly powerful mixing, if

$$\varphi(\tau)=\operatorname{vrai\,sup}_{\omega\in\Omega}\ \sup_{A\in\mathfrak{A}(t+\tau,\infty)}|\mathsf{P}\{A|\mathfrak{A}(-\infty,t)\}-\mathsf{P}\{A\}|\xrightarrow[\tau\to\infty]{}0. \quad (1.5)$$

It is not difficult to check that

$$\varphi(\tau)=\sup_{A\in\mathfrak{A}(t+\tau,\infty),\ B\in\mathfrak{A}(-\infty,t)}\frac{|\mathsf{P}(AB)-\mathsf{P}(A)\mathsf{P}(B)|}{\mathsf{P}(A)}.$$

One Additional Determination of regularity is connected with the concept of quantity of information.

Let $\{\xi_1(t),\ t\in T\}$ and $\{\xi_2(s),\ s\in S\}$ two arbitrary random processes (two families of random variables). Let us designate by $\mathfrak{A}_1$ and $\mathfrak{A}_2$ minimum σ-algebra of events those which were generated respectively by families $\{\xi_1(t),\ t\in T\}$, $\{\xi_2(s),\ s\in S\}$. Quantity of information relative to random process $\{\xi_1(t),\ t\in T\}$, that included in process $\{\xi_2(s),\ s\in S\}$, is called value

$$I(\xi_1,\ \xi_2)=\sup\sum\mathsf{P}(A_iB_j)\ln\frac{\mathsf{P}(A_iB_j)}{\mathsf{P}(A_i)\mathsf{P}(B_j)}, \quad (1.6)$$

where sup it is taken on all possible final decompositions of space

of elementary issues Ω into the nonintersecting events $(A_1, A_2, \ldots, A_n)$, $(B_1, B_2, \ldots, B_m)$, whereupon $A_i \in \mathfrak{A}_1$, a $B_j \in \mathfrak{A}_2$. It is not difficult to comprehend that $I(\xi_1, \xi_2) = I(\xi_2, \xi_1)$. Applying Jensen's inequality to the convex function $x \ln x$, $x > 0$, it is possible to show that I $(\xi_1, \xi_2) \geq 0$, whereupon I $(\xi_1, \xi_2) = 0$ when and only when σ-algebra $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are independent [1].

FOOTNOTE [1]. See, for example, [19]. ENDFOOTNOTE.

Computing the quantity of information, which is contained in the past $\{\xi(s), s \leqslant t\}$ of our process relative to the future $\{\xi(s), s \geqslant t+\tau\}$, let us arrive at the following determination.

5. Stationary process ξ (t) is called informationally regular, if

$$I(\tau) = I\{(\xi(s), s \leqslant t), (\xi(s), s \geqslant t+\tau)\} \xrightarrow[\tau\to\infty]{} 0. \quad (1.7)$$

Value I (τ) occasionally referred to as the informational coefficient of regularity.

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Further, the left side of equality (1.1) can be registered in

the form $M\eta_1\eta_2$, where $\eta_1 = \chi_A - P\{A\}$, $\eta_2 = \chi_B - P\{B\}$. If we examine here arbitrary random variables η_1, η_2, provided would make sense mathematical expectation $M\eta_1\eta_2$, we let us arrive at the following concept:

the maximum correlation coefficient between the systems of random variables $\{\xi_1(t), t \in T\}$ and $\{\xi_2(s), s \in S\}$ is called value

$$r(\xi_1, \xi_2) = \sup M\eta_1\eta_2,$$

where sup it is taken according to all η_1, η_2, measurable respectively relative to σ-algebra $\mathfrak{A}_1, \mathfrak{A}_2$ and by such, that

$$M\eta_1 = M\eta_2 = 0, \quad M|\eta_1|^2 = M|\eta_2|^2 = 1.$$

This concept has simple geometric meaning. Let us designate H the set of all random variables η with the final mathematical expectation $M|\eta|^2$; H there is hilbert space with scalar product $(\eta_1, \eta_2) = M\eta_1\eta_2$.

1 If H_1 and H_2 - the subspaces H, which consist of the random variables, measurable respectively relatively $\mathfrak{A}_1$ and $\mathfrak{A}_2$, that r (ξ_1, ξ_2) are a cosine of the minimum angle between H_1 and H_2 (comp. below from 2°).

The concept of the maximum correlation coefficient leads for a stationary process ξ (t) to the following condition of full/total/complete regularity.

6.

$$r(\tau) = r((\xi(s),\ s \leqslant t),\ (\xi(s),\ s \geqslant t+\tau)) \xrightarrow[\tau\to\infty]{} 0. \quad (1.8)$$

Under condition (1.8) let us call random process ξ (t) completely regular.

Is obvious, always r (τ) $\geq$ α (τ). However, for Gaussian stationary processes, as is known [1]

$$\alpha(\tau) \leqslant r(\tau) \leqslant 2\pi\alpha(\tau), \quad (1.9)$$

and, thus,

Gaussian stationary process ξ (t) satisfies the condition of powerful mixing then and only then when it is completely regular.

FOOTNOTE [1]. See [22], page 249. ENDFOOTNOTE.

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Being limited to Gaussian processes - by the basic subject of

research of this book, it is logical to attempt to express conditions 1-6 in spectral terms, since in the Gaussian case the spectral or correlation function completely determines process. It is understandable that in this case conditions themselves 1-6 must obtain some equivalent formulation in the terms of spaces H (T), and, it goes without saying, any this formulation will make sense for arbitrary stationary in the broad sense processes. Therefore we will give some natural analogs of conditions 1-6 for the processes, stationary in the broad sense. For such processes the role of independence plays orthogonality, instead of σ-algebra $\mathfrak{A}(T)$ are examined spaces H (T), and the operator of the taking of conditional mathematical expectation $M\{\cdot|\mathfrak{A}(T)\}$ is substituted by the operator of design $\hat{M}\{\cdot|H(T)\}$ (in hilbert space H) to subspace H (T).

Let $\xi(t)$, $M\xi(t) = 0$, be the random process, stationary in the broad sense.

1°. Process ξ (t) is called (linearly) regular, if the space

$$H(-\infty, -\infty) = H(-\infty) \stackrel{\text{def}}{=} \bigcap_t H(-\infty, t)$$

is trivial, i.e., it consists only of random variables, with probability 1 equal to 0.

It is analogous to (with 1.1), process ξ (t) is linearly regular in that and only that case, if

$$\| \hat{M} \{\eta \mid H(-\infty, -\tau)\} \| =$$
$$= M^{1/2} \{(\hat{M} \{\eta \mid H(-\infty, -\tau)\})^2\} \xrightarrow[\tau \to \infty]{} 0. \quad (1.10)$$

It is clear that if ξ (t) is regular in the sense of definition 1, then it is linearly regular.

The condition of linear regularity plays the very important role in the theory of the forecast/prediction of stationary random processes.

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one of the central results of this theory is the fact that the stationary in the broad sense process ξ (t) is linearly regular when and only when it allow/assumes the representation of Wold [1]:

$$\xi(t) = \sum_{-\infty}^{t} c(t-s)\,\zeta(s) \quad \left(\sum_{0}^{\infty} |c(t)|^2 < \infty\right) \quad (1.11)$$

with discrete t, and

$$\xi(t) = \int_{-\infty}^{t} c(t-s)\,\zeta(ds) \quad \left(\int_{0}^{\infty} |c(t)|^2 dt < \infty\right) \quad (1.12)$$

with continuous t, where $\zeta(ds)$ there is a stochastic orthogonal measure on the time/temporary axis: ($M\zeta(\Delta_1)\zeta(\Delta_2) = |\Delta_1 \cap \Delta_2|$ ($|\Delta|$ indicates the

length of time interval Δ).

FOOTNOTE [1]. See, for example, [22], page 77, 162. ENDFOOTNOTE.

From formulas (1.11), (1.12) it follows that σ-algebra $\mathfrak{A}(-\infty, t)$ is contained in σ-algebra $\mathfrak{B}(-\infty, t)$, generated by values $\zeta(\Delta)$ on time/temporary semi-axis (-∞, t), and

$$\bigcap_t \mathfrak{A}(-\infty, t) \subseteq \bigcap_t \mathfrak{B}(-\infty, t).$$

In the Gaussian case the orthogonality is equivalent independence and $\zeta(ds)$ is a stochastic measure with independent values (specifically, for discrete t we deal with the sequence of independent Gaussian values $\zeta(s), s=t, t-1, \ldots$). In this case according to known law "0 or 1" the intersection $\bigcap_t \mathfrak{B}(-\infty, t)$ is trivial, and therefore from the linear regularity of Gaussian stationary process ξ (t) escape/ensues its regularity (in the sense of definition 1). Thus occurs the following proposition:

The Gaussian stationary process ξ (t) is regular when and only when it is linearly regular.

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Linearly regular stationary processes continually are described. Specifically,

Stationary process ξ (t) is linearly regular when and only when its spectral measure is absolutely continuous, and spectral density f (λ) satisfies the condition:

$$\int_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda > -\infty \quad (1.13)$$

with discrete t, and

$$\int_{-\infty}^{\infty} \frac{\ln f(\lambda)}{1+\lambda^2}\, d\lambda > -\infty \quad (1.14)$$

with continuous t (see in regard to this [22], pages 85, 161 or §2 of Chapter II).

Further, analogous with that, as this was done during passage of (1.1) to (1.2), it is possible to take in (1.10) sup on η, and we will arrive at the following condition.

2°. Process ξ (t) is called completely (linearly) regular, if

$$\rho(\tau) = \sup_{\eta \in H(0,\infty),\ \|\eta\|=1} \|\hat{M}\{\eta \mid H(-\infty, -\tau)\}\| \xrightarrow[\tau\to\infty]{} 0. \quad (1.15)$$

Value ρ (τ) let us call the coefficient of full/total/complete

(linear) regularity.

Is obvious, always $\rho(\tau) \leq r(\tau)$. But in the case of Gaussian processes occurs the following fundamental for our further research fact [1]:

$$\rho(\tau)=r(\tau). \tag{1.16}$$

FOOTNOTE [1]. See [22], page 249. ENDFOOTNOTE.

This indicates that

the Gaussian stationary process $\xi(t)$ is completely regular (in the sense of definition 6) when and only when it is completely is linearly regular.

It is easy to see that

$$\rho(\tau)=\sup|M\eta_1\bar{\eta}_2|=\sup|(\eta_1,\eta_2)|,$$

where sup is taken on all $\eta_1 \in H(-\infty, 0)$, $\eta_2 \in H(\tau, \infty)$ with $M|\eta_1|^2 = M|\eta_2|^2 = 1$, so that $\rho(\tau)$ there is a cosine of the minimum angle between subspaces $H(-\infty, 0)$, $H(\tau, \infty)$. Let us note that the condition $\rho(\tau) < 1$ is more powerful than (linear) regularity.

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Further, besides the minimum angle between subspaces H (-∞, t), H (t + τ, ∞) it is possible to introduce other similar characteristics of these subspaces, being based on the following idea.

Let us designate by $\mathscr{P}_\tau^-(=\mathscr{P}_\tau)$ the operator of design in H on H (-∞,-τ), by $\mathscr{P}_\tau^+$ - projector on H (τ, ∞) let us introduce the operators

$$B_\tau^- = B_\tau = \mathscr{P}_\tau^- \mathscr{P}_0^+ \mathscr{P}_\tau^-, \quad B_\tau^+ = \mathscr{P}_0^+ \mathscr{P}_\tau^- \mathscr{P}_0^+, \quad \tau \geqslant 0.$$

Subspaces H (-∞, 0), H (τ, ∞) are orthogonal in that and only that case, if $B_\tau^\pm = 0$. This fact suggests to examine as the conditions of regularity the convergence of the operators B_τ to 0 with $\tau \to \infty$ for one or the other (uniform, gil'berta - Schmidt, nuclear) topology; above we will see, that thus possible to reformulate all the enumerated above conditions of regularity.

The conditions of regularity, expressed in the terms of hilbert spaces H (T), allow/assume analytical formulation, since is an isometric conformity $\xi(t) \leftrightarrow e^{i\lambda t}$ with the described in chapter II subspaces $L_T(F)$. Moreover, in the case of Gaussian processes this concerns other conditions (since, for example, $\rho(\tau) \asymp r(\tau) \asymp \alpha(\tau)$). The

translation/conversion of the enumerated above conditions of regularity for analytical language and the further solution of the appearing analytical problems for the final target/purpose of obtaining the criteria of regularity, expressed through spectral characteristics, and composes the basic content of this and two subsequent chapters.

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§2. Conditions of regularity and the operators B_τ

Let ξ (t) be a stationary Gaussian process. Let us consider operators [1] $B_\tau^-(=B_\tau)$ and B_τ^+, introduced above in §1.

FOOTNOTE [1]. Into the theory of random processes the operators B_τ are introduced in the work of I. M. Gelfand and A. M. Yaglom "On the calculation of quantity of information about the random function, which is contained into another such functions (P.M.S. XII, V. 1 (1957)). Communication/connection of the operators B_τ with the conditions of regularity noted Yaglom (A. M. Yaglom, Stationary Gaussian processes satisfying the strong mixing condition and best

predictable functionals, Bernoully - Bayes - Laplace, Anniv. volume Springer-Verlag, 1965). ENDFOOTNOTE.

These are the positive self-adjoint operators. Let us show that all conditions of regularity, formulated in §1 for a Gaussian process ξ (t), can be expressed in the terms of convergence $B_\tau \to 0$ with $\tau \to \infty$ - (in this or another sense).

Theorem 1. Stationary Gaussian process ξ (t) is regular when and only when with $\tau \to \infty$ the operators B_τ weakly converge to 0, i.e., if for any random variable $\eta \in H(-\infty, \infty)$

$$\| B_\tau \eta \| = \mathsf{M}^{1/2} | B_\tau \eta |^2 \xrightarrow[\tau \to \infty]{} 0.$$

In accordance with what has been said in §1 it is possible not to distinguish regularity and linear regularity. Therefore theorem 1 immediately follows from the determination of linear regularity and inequality

$$\| B_\tau \eta \| \leqslant \| \mathscr{P}_\tau \eta \|.$$

Theorem 2. In order that the stationary Gaussian process ξ (t) would be completely regular, it is necessary and sufficient in order that the operators B_τ converge to zero evenly, moreover, the coefficient of regularity is $\rho(\tau) = \| B_\tau \|$.

It goes without saying that this theorem remains valid for arbitrary stationary processes, if we keep in mind linearly completely regular processes. For Gaussian processes on the strength of relationship/ratios (1.9), (1.16) this theorem simultaneously gives the criterion of powerful mixing.

Theorem 2 immediately follows from the determination of the coefficient of regularity ρ (τ). Actually, regarding

$$\rho(\tau) = \sup_{\eta \in H(0,\infty),\ \|\eta\|=1} \|\mathcal{P}_\tau \eta\| = \sup\left(\mathcal{P}_\tau \mathcal{P}_0^+ \eta,\ \mathcal{P}_0^+ \eta\right)^{1/2} =$$
$$= \sup\left(B_\tau^+ \eta,\ \mathcal{P}_0^+ \eta\right) = \sup_{\substack{\eta \in H(-\infty,\infty),\\ \|\eta\|=1}} \left(B_\tau^- \eta,\ \eta\right) = \|B_\tau^-\| = \|B_\tau\|.$$

For processes with discrete time it is possible to demonstrate the more interesting version of this theorem.

Theorem 3. Stationary process ξ (t) with discrete time is completely regular when and only when it is regular, and operator B_1 is completely continuous.

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Lemma 1. Let H_1 and H_2 the essence of the subspace separable be hilbert space H. Let us designate by $\mathcal{P}_1, \mathcal{P}_2$ the operators of design in H respectively on H_1 and H_2. Let, further, $B_1 = \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_1$, $B_2 = \mathcal{P}_2 \mathcal{P}_1 \mathcal{P}_2$. If

the operators B_1, B_2 are completely continuous, in H_1 it is possible to select orthogonal base $\{\{e_{1j}\}\oplus\{e'_{1k}\}\}$, ~~and~~ in H_2 - orthogonal base $\{\{e_{2j}\}\oplus\{e'_{2k}\}\}$ with the following properties:

1) all vectors e'_{1k} are orthogonal H_2, all vectors e'_{2k} orthogonal H_1;

2) scalar products (e_{1i}, e_{2j}) are different from zero, only if i = j;

3) e_{1j} - the eigenvectors of operator B_1, e_{2j} - operator B_2.

Proof of lemma. From the elementary properties of projection operators $\mathcal{P}_i$: $\mathcal{P}_i^2=\mathcal{P}_i$, $\mathcal{P}_i^*=\mathcal{P}_i$ easily it is derive/concluded, what B_1 and B_2 - positive self-adjoint operators, simultaneously completely continuous or not. Completely continuous self-adjoint operator B_1 possesses the full/total/complete in the range of its values sequence of the orthogonal standardized/normalized eigenvectors e_{11}, e_{12}, ..., that correspond to different from zero to eigenvalues λ_1, λ_2, ... (completeness means that any vector $\varphi=B_1h$, orthogonal to all vectors e_{1j}, is equal to zero) [1].

FOOTNOTE [1]. See, for example, [2], page 189. ENDFOOTNOTE.

Let us assume $e_{2l}=\frac{\mathcal{P}_2 e_{1l}}{\|\mathcal{P}_2 e_{1l}\|}=\frac{1}{\sqrt{\lambda_l}}\mathcal{P}_2 e_{1l}$. It is easy to see that whole e_{2l} essence the eigenvectors of operator B_2, which correspond to eigenvalues λ_l. Actually,

$$B_2 e_{2l}=\frac{1}{\sqrt{\lambda_1}}\mathcal{P}_2\mathcal{P}_1\mathcal{P}_2 e_{1l}=\frac{1}{\sqrt{\lambda_1}}\mathcal{P}_2 B_1 e_{1l}=\lambda_1 e_{2l}.$$

By examining projections $\mathcal{P}_1 e_{2l}$, let us ascertain that vectors $\{e_{2l}\}$ are contained by all eigenvectors of operator B_2.

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Let now the orthonormalized vectors $\{e'_{1k}\}$ $(\{e'_{2k}\})$ complement of system $\{e_{1l}\}$ $(\{e_{2l}\})$ to base in H_1 (in H_2). Let us demonstrate that the bases $\{\{e_{1l}\}\oplus\{e'_{1k}\}\}$ and $\{\{e_{2l}\}\oplus\{e'_{2k}\}\}$ possess the necessary properties.

1) Let us demonstrate that all vectors e'_{1k} are orthogonal H_2, i.e., that with all k $\mathcal{P}_2 e'_{1k}=0$. Is really/actually

$$\|\mathcal{P}_2 e'_{1k}\|^2=(e'_{1k},\ \mathcal{P}_2 e'_{1k})=(\mathcal{P}_1 e'_{1k},\ \mathcal{P}_2\mathcal{P}_1 e'_{1k})=(e'_{1k},\ B_1 e'_{1k})=0,$$

since on the strength of completeness vector $B_1 e'_{1k}$, orthogonal to all vectors e_{1l}, is equal to zero. Analogously it is proven, that with all to k $e'_{2k}\perp H_1$.

2) Let us count scalar products (e_{1i}, e_{2j}). We have

$$\sqrt{\lambda_1}(e_{1i}, e_{2j}) = (\mathscr{P}_1 e_{1i}, \mathscr{P}_2 e_{1j}) = (\mathscr{P}_2 \mathscr{P}_1 e_{1i}, e_{1j}) = \\ = (B_1 e_{1i}, e_{1j}) = \lambda_1 (e_{1i}, e_{1j}) = \lambda_1 \delta_{ij},$$

where δ_{ij} - Kronecker's symbol. Lemma is demonstrated.

Proof of theorem 3. Suppose that operator B_1 is completely continuous. Let $e_1, e_2, \dots$ - its standardized/normalized eigenvectors, and $\lambda_1 \geq \lambda_2 \geq \dots$ - the corresponding to them eigenvalues. On the basis of lemma 1 addition $e_1, e_2, \dots$ to base in $H(-\infty, -1)$ orthogonally to $H(0, \infty)$; therefore without loss of generality it is possible to count that already vectors $e_1, e_2, \dots$ form base in $H(-\infty, -1)$.

We will take arbitrary cell/element $\eta \in H(-\infty, -\tau)$ let us register it in the form

$$\eta = \sum_1^\infty a_j e_j,$$

where

$$|a_j| = |(\eta, e_j)| = |(\eta, \mathscr{P}_\tau e_j)| \leqslant \|\mathscr{P}_\tau e_j\|.$$

On the strength of regularity for any that which was fix/recorded j $\left(\|\mathscr{P}_\tau e_j\| \xrightarrow[\tau\to\infty]{} 0.\right)$ Consequently

$$\| B_\tau \| = \sup_{\|\eta\|=1} \|\mathscr{P}_0^+ \eta\| = \sup_{\|\eta\|=1} \Big(\sum_{i,l} a_i \bar{a}_l (\mathscr{P}_0^+ e_i, \mathscr{P}_0^+ e_l)\Big)^{1/2} = \\ = \sup_{\|\eta\|=1} \Big(\sum_{i,l} a_i a_l (B_1 e_i, e_l)\Big)^{1/2} = \sup_{\|\eta\|=1} \Big(\sum_1^\infty |a_l|^2 \lambda_l\Big)^{1/2} \leqslant \\ \leqslant \lambda_1 s \max_{1 \leqslant l \leqslant s} \|\mathscr{P}_\tau e_l\|^2 + \lambda_{s+1} = \lambda_{s+1} + o(1).$$

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The last/latter expression can be done as by conveniently small, taking into account that $\lambda_s = 0$.

Let us demonstrate need. Let us note first that it suffices to demonstrate operator's full/total/complete continuity B_1 on H (-∞, -1). We will write space H (-∞, -1) in the form $H(-\infty, -\tau) \oplus R_\tau$, where orthogonal complement R_τ space H (-∞,-τ) to H (-∞, -1) is finite-dimensional. If we designate Q_τ projector in H (-∞, -1) on R_τ, then $\mathscr{P}_\tau + Q_\tau$ there is a single operator on H (-∞, -1). Consequently

$$B_1 = (\mathscr{P}_\tau + Q_\tau) B_1 (\mathscr{P}_\tau + Q_\tau) = \mathscr{P}_\tau B_1 \mathscr{P}_\tau + K_\tau = B_\tau + K_\tau,$$

where the operator K_τ is finite. But then

$$\| B_1 - K_\tau \| = \| B_\tau \| = \rho(\tau) \xrightarrow[\tau \to \infty]{} 0.$$

This means that operator B_1 allow/assumes as good as suitable

approach/approximation by the finite-dimensional operators and, which means, is completely continuous [1].

FOOTNOTE [1]. See, for example, [5], page 47. ENDFOOTNOTE.

Observation. Actually, proving the sufficiency of the conditions of theorem, we demonstrated more common/general/total result, namely:

Let $\xi(t)$ - stationary Gaussian process with discrete or continuous time, and let for a given $\tau > 0$ operator B_τ is completely continuous; then process $\xi(t)$ is completely regular.

Further let us demonstrate the following proposition.

Theorem 4. In order that the stationary Gaussian process would be absolutely regular, it is necessary and sufficient in order that it would be regular and for a given τ_0 operator B_{τ_0} would be completely continuous operator with the final trace [2].

FOOTNOTE [2]. I. e., nuclear. ENDFOOTNOTE.

In this case and all operators $B_\tau, \tau \geqslant \tau_0$, have the final trace, but the coefficient (absolute) of regularity are

$$\beta(\tau) \asymp \sqrt{\mathrm{Sp}\, B_\tau}, \qquad \tau \to 0, \qquad (2.1)$$

where $\mathrm{Sp}\, B_\tau$ it indicates operator's trace B_τ.

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It is more precise, if $\beta(\tau) \to 0$,

$$\frac{1}{2\pi} \leqslant \varliminf_{\tau} \frac{\beta(\tau)}{\sqrt{\mathrm{Sp}\, B_\tau}} \leqslant \varlimsup_{\tau} \frac{\beta(\tau)}{\sqrt{\mathrm{Sp}\, B_\tau}} \leqslant \frac{1}{2\sqrt{2\pi}}.$$

Proof [1].

FOOTNOTE [1]. In proof are used some results from the article of V. A. Volkonskiy and ~~B.~~ Yu. A. Rozanova "some limit theorems for random functions. II", theoretical prob. and its applications. VI, No 2 (1961), 202-215. ENDFOOTNOTE.

Preliminarily let us establish/install several common properties of coefficient β (τ). Let $\{\eta_1(u),\ u \in U;\ \eta_2(v),\ v \in V\}$ be an arbitrary system of

Gaussian random variables. Let us designate by $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}$ σ-algebra of events, generated respectively by the systems of values $\{\eta_1(u),\ u \in U\}$, $\{\eta_2(v),\ v \in V\}$, $\{\eta_1(u),\ \eta_2(v);\ u \in U,\ v \in V\}$. Let, further, H, H_1, H_2 - the linear spaces, stretched respectively on $\{\eta_1, \eta_2\}$, $\{\eta_1\}$, $\{\eta_2\}$. Let us designate by $\hat{\mathscr{P}}_1$, $\hat{\mathscr{P}}_2$ projectors in H on H_1, H_2. Let; finally,

$$\hat{B}_1 = \hat{\mathscr{P}}_1\hat{\mathscr{P}}_2\hat{\mathscr{P}}_1, \quad \hat{B}_2 = \hat{\mathscr{P}}_2\hat{\mathscr{P}}_1\hat{\mathscr{P}}_2.$$

Let us assume, further,

$$\beta = \beta(\{\eta_1(u),\ \eta_2(v)\}) = \mathsf{M} \sup_{A \in \mathfrak{A}_2} |\mathsf{P}\{A \mid \mathfrak{A}_1\} - \mathsf{P}\{A\}|.$$

Let us designate by Q_1, Q_2, Q the probability measures, generated on algebras $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}$ by random variables $\{\eta_1(u),\ u \in U;\ \eta_2(v),\ v \in V\}$. By $\tilde{Q}$ let us designate probability measure on $\mathfrak{A}$, that coincides on $\mathfrak{A}_1$ with measure Q_1, on $\mathfrak{A}_2$ - with measure Q_2 and such, that the algebras $\mathfrak{A}_1, \mathfrak{A}_2$ are independent relative to measure Q (it is possible to count $\tilde{Q} = Q_1 \times Q_2$).

Lemma 2. Occurs the equality

$$\beta = \frac{1}{2} \operatorname{Var}(Q - \tilde{Q})$$

Proof of lemma. Let us demonstrate first that

$$\frac{1}{2} \operatorname{Var}(Q - \tilde{Q}) \leqslant \beta. \qquad (2.2)$$

It is sufficient to show that for all events C of form $C = \bigcup_i A_i B_i$, where $A_i \in \mathfrak{A}_1,\ B_i \in \mathfrak{A}_2$ and everything $A_i B_i$ are incompatible/inconsistent in pairs,

occurs the inequality

$$|Q(C)-\tilde{Q}(C)|\leqslant\beta.$$

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If C - the event of the form indicated, it always it is possible to register in the form $C=\bigcup_i A_iB_i'$, where this time any two events B_i', B_j' either are incompatible/inconsistent or coincide (all events A_iB_i' are incompatible/inconsistent). Let us agree instead of B_i' to write again B_i. We have, counting $Q(C)>\tilde{Q}(C)$,

$$Q(C)-\tilde{Q}(C)=\sum_i [\mathsf{P}\{A_iB_i\}-\mathsf{P}\{A_i\}\mathsf{P}\{B_i\}]= \\ =\sum_i \int_{B_i} [\mathsf{P}\{A_i\,|\,\mathfrak{A}_2\}-\mathsf{P}\{A_i\}]\,d\mathsf{P}. \quad (2.3)$$

If $B_{i_1},\ldots,B_{i_l}$ are any coinciding events from sequence $B_1, B_2, \ldots$, then the corresponding to them events A_{i_s} do not intersect, and, set/assuming $\tilde{A}_i=\bigcup_{s=1}^{l} A_{i_s}$, we will find that

$$\sum_s \int_{B_{i_s}} [\mathsf{P}\{A_{i_s}\,|\,\mathfrak{A}_2\}-\mathsf{P}\{A_{i_s}\}]\,d\mathsf{P}=\int_{B_{i_1}} [\mathsf{P}\{\tilde{A}_i\,|\,\mathfrak{A}_2\}-\mathsf{P}\{\tilde{A}_i\}]\,d\mathsf{P}. \quad (2.4)$$

Hence and from (2.3) it follows that

$$Q(C)-\tilde{Q}(C)=\sum_i \int_{\tilde{B}_i} [\mathsf{P}\{\tilde{A}_i\,|\,\mathfrak{A}_2\}-\mathsf{P}\{\tilde{A}_i\}]\,d\mathsf{P},$$

where $\tilde{B}_1, \tilde{B}_2, \ldots$ - the sequence of nonintersecting events from $\mathfrak{A}$.

Consequently,

$$|Q(C)-\tilde{Q}(C)| \leqslant \sum_i \int_{\tilde{B}_i} \sup_{A\in\mathfrak{A}_1} |P\{A\,|\,\mathfrak{A}_2\} - P\{A\}|\,dP = \\ = \int_{\cup \tilde{B}_i} \sup_{A\in\mathfrak{A}_1} |P\{A\,|\,\mathfrak{A}_2\} - P\{A\}|\,dP \leqslant \beta.$$

Inequality (2.2) is proved.

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We will proceed to the proof of reverse inequality. Let us designate by $\tilde{\mathfrak{A}}_1$ the subset $\mathfrak{A}_1$, which consists of all events of form $\{(\eta_1(u_1), \ldots, \eta_1(u_s)) \in \Lambda\}$, where Λ is an association of 2s-graduated cubes with rational apex/vertexes. Many events $\tilde{\mathfrak{A}}_1$ is calculating. It is possible to count which during determination β sup is taken only on $A\in\tilde{\mathfrak{A}}_1$(at the worst reader can include/connect this requirement in determination β). Each simple event $\omega\in\Omega$ answers event $A_l\in\tilde{\mathfrak{A}}$ such, that

$$P\{A_l\,|\,\mathfrak{A}_2\} - P\{A_l\} \geqslant \sup_{A\in\mathfrak{A}_1} |P\{A\,|\,\mathfrak{A}_2\} - P\{A\}| - \varepsilon,$$

where $\varepsilon>0$. Let us designate by B_l the events, which consist of all ω, that correspond to one A_l. It is obvious, $B_l\in\mathfrak{A}_2$. It is possible to count that all events B_l are incompatible/inconsistent (otherwise we

we could consider events $\tilde{B}_1=B_1, \tilde{B}_2=B_2 \setminus B_1, \tilde{B}_3 \setminus (B_2 \cup B_1), \ldots$). Consequently, are incompatible/inconsistent events $A_j B_j$. Therefore

$$\beta=\int_{\Omega} \sup_{A\in\mathfrak{A}_1} |\mathsf{P}\{A\,|\,\mathfrak{A}_2\}-\mathsf{P}\{A\}\,|\,d\mathsf{P}\leqslant$$
$$\leqslant\sum_j \int_{B_j} [\mathsf{P}\{A_j\,|\,\mathfrak{A}_2\}-\mathsf{P}\{A_j\}]\,d\mathsf{P}+\varepsilon=$$
$$=\sum_j [\mathsf{P}\{A_jB_j\}-\mathsf{P}\{A_j\}\mathsf{P}\{B_j\}]+\varepsilon=$$
$$=[Q(\cup A_jB_j)-\tilde{Q}(\cup A_jB_j)]+\varepsilon\leqslant\frac{1}{2}\operatorname{Var}(Q-\tilde{Q})+\varepsilon.$$

Lemma is demonstrated.

Lemma 3. If sets U and V are final, they occur of the inequality

$$\frac{1}{2\pi}\sqrt{\operatorname{Sp} B_1}-\frac{1}{2}\left(2\operatorname{Sp} B_1+\frac{1}{2}\,\frac{3\operatorname{Sp} B_1+\frac{1}{2}(\operatorname{Sp} B_1)^2}{(1-\|B_1\|)^2}\right)\leqslant$$
$$\leqslant\beta\leqslant\frac{1}{2\sqrt{2\pi}}\sqrt{\operatorname{Sp} B_1}+\frac{1}{2}\left(2\operatorname{Sp} B_1+\frac{1}{2}\,\frac{3\operatorname{Sp} B_1+\frac{1}{2}(\operatorname{Sp} B_1)^2}{(1-\|B_1\|)^2}\right). \tag{2.5}$$

Constants $\frac{1}{2\sqrt{2\pi}}$ and $\frac{1}{2\pi}$ with $\sqrt{\operatorname{Sp} B_1}$ precise.

end section.

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Proof. On the basis of lemma 1 it is possible to count that $\{\eta_1(u), u \in U\} = \{\eta_{11}, \ldots, \eta_{1n}\}, \{\eta_2(v), v \in V\} = \{\eta_{21}, \ldots, \eta_{2n}\}$, the vectors $\{\eta_{11}, \ldots, \eta_{1n}\}, \{\eta_{21}, \ldots, \eta_{2n}\}$ consist of independent normal values with the average of zero and dispersion 1, with

$$M\eta_{1i}\eta_{2i} = \delta_{ij}\rho_i, \qquad 1 \geqslant \rho_1 \geqslant \ldots \geqslant \rho_n > 0.$$

In this case

$$\mathrm{Sp}\, B_1 = \mathrm{Sp}\, B_2 = \sum \rho_i^2, \quad \|B_1\| = \|B_2\| = \rho_1^2.$$

In the case $\rho_1 = 1$ assertion of lemma is trivial; therefore we will count $\rho_1 < 1$. In the done assumptions, by designating through $p_1(x)$, $p_2(y)$, $p_{12}(x, y)$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ the densities of distribution of Gaussian vectors $(\eta_{11}, \ldots, \eta_{1n})$, $(\eta_{21}, \ldots, \eta_{2n})$, $(\eta_{11}, \ldots, \eta_{1n}, \eta_{21}, \ldots, \eta_{2n})$, let us have

$$\mathrm{Var}\,[Q - \tilde{Q}] = \int_{R^n}\int_{R^n} |\, p_{12}(x, y) - p_1(x)\, p_2(y)\,|\, dx\, dy, \quad (2.6)$$

$$p_1(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\sum x_i^2}, \quad p_2(y) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\sum y_i^2},$$

$$p_{12}(x, y) = (2\pi)^{-n} \prod \frac{1}{\sqrt{1-\rho_i}} \exp\left\{ -\frac{x_i^2 + y_i^2 - 2\rho_i x_i y_i}{2(1-\rho_i^2)} \right\}.$$

let us rewrite right side (2.6) in the form

$$\int_{R^n}\int_{R^n}\left|\frac{p_{12}(x, y)}{p_1(x)\,p_2(y)}-1\right|p_1(x)\,p_2(y)\,dx\,dy=\tilde{\mathrm{M}}\,|\,e^{-\zeta}-1\,|,$$

where

$$\zeta=-\ln\frac{p_{12}(\eta_1,\eta_2)}{p_1(\eta_1)\,p_2(\eta_2)}=\\=\sum\frac{\rho_i^2\eta_{1i}^2+\rho_i^2\eta_{2i}^2-2\rho_i\eta_{1i}\eta_{2i}}{2(1-\rho_i^2)}+\frac{1}{2}\ln(1-\rho_i^2),$$

a mathematical expectation $\tilde{\mathrm{M}}$ it is computed relative to measure $\tilde{Q}$, i.e., on the assumption that all values $(\eta_{11},\ldots,\eta_{2n})$ are independent.

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Is decomposed $e^{-\zeta}$ by Taylor's formula with remainder in the form of Lagrange:

$$e^{-\zeta}=1-\zeta+\frac{\zeta^2}{2}e^{-\theta\zeta},\quad 0<\theta<1.$$

Then

$$\begin{aligned}\tilde{M}|\zeta^{-\zeta}-1|=&\\ =\int_{\{\zeta\geqslant 0\}}\left|\zeta-\frac{\zeta^2}{2}e^{-\theta\zeta}\right|d\tilde{Q}+&\int_{\{\zeta<0\}}\left|\zeta-\frac{\zeta^2}{2}e^{-\theta\zeta}\right|d\tilde{Q}\geqslant\\ \geqslant\int_{\{\zeta\geqslant 0\}}\left(|\zeta|-\frac{\zeta^2}{2}\right)d\tilde{Q}+&\int_{\{\zeta<0\}}\left(|\zeta|-\frac{\zeta^2}{2}e^{-\zeta}\right)d\tilde{Q}\geqslant\\ &\geqslant\tilde{M}|\zeta|-\frac{1}{2}\tilde{M}\zeta^2-\frac{1}{2}\tilde{M}\zeta^2e^{-\zeta}.\end{aligned}$$

Analogously is derive/concluded the inequality

$$\tilde{M}|e^{-\zeta}-1|\leqslant\tilde{M}|\zeta|-\frac{1}{2}\tilde{M}\zeta^2-\frac{1}{2}\tilde{M}\zeta^2e^{-\zeta}.\qquad(2.7)$$

Noting that $\tilde{M}\zeta^2e^{-\zeta}=M\zeta^2$, where M indicates mathematical expectation relative to Q, let us have

$$\tilde{M}|\zeta|-\frac{1}{2}(\tilde{M}\zeta^2+M\zeta^2)\leqslant 2\beta\leqslant\tilde{M}|\zeta|+\frac{1}{2}(\tilde{M}\zeta^2+M\zeta^2).\qquad(2.8)$$

Let us introduce new random variables u_i, v_i, by set/assuming

$$u_i=\frac{1}{\sqrt{2}}(\eta_{1i}-\eta_{2i}),\quad v_i=\frac{1}{\sqrt{2}}(\eta_{1i}+\eta_{2i}).$$

It is obvious, $\tilde{M}u_iv_i=Mu_iv_i=0$, so that Gaussian random variables $(u_1,\ldots,u_n,$

$v_1, \ldots, v_n)$ are independent relative to both distributions $\tilde{Q}$ and Q. Furthermore,

$$\tilde{M}u_i = \tilde{M}v_i = 0, \quad \tilde{M}u_i^2 = \tilde{M}v_i^2 = 1,$$
$$Mu_i = Mv_i = 0, \quad Mu_i^2 = \left(1 - \frac{\rho_i}{2}\right), \quad Mv_i^2 = \left(1 + \frac{\rho_i}{2}\right).$$

In the new variables

$$\zeta = \frac{1}{2}\sum\left(\frac{\rho_i(1+\rho_i)u_i^2 - \rho_i(1-\rho_i)v_i^2}{1-\rho_i^2} + \ln(1-\rho_i^2)\right).$$

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Hence easily it is obtained, that

$$\tilde{M}\zeta^2 = \tilde{D}\zeta + (\tilde{M}\zeta)^2 \leqslant$$
$$\leqslant \frac{1}{4}\left[2\,\mathrm{Sp}\,B_1 + \frac{2\,\mathrm{Sp}\,B_1}{(1-\|B_1\|)^2} + \left(\mathrm{Sp}\,B_1\,\frac{\|B_1\|^2}{1-\|B_1\|}\right)^2\right],$$
$$M\zeta^2 \leqslant \frac{1}{4}\left[2\,\mathrm{Sp}\,B_1 + \frac{2\,\mathrm{Sp}\,B_1}{(1-\|B_1\|)^2} + \left(\frac{\mathrm{Sp}\,B_1}{1-\|B_1\|}\right)^2\right].$$

Somewhat more complexly to obtain estimation for $\tilde{M}|\zeta|$. We first will replace ζ with the simpler random variable

$$\zeta_1 = \frac{1}{2}\sum\rho_i(u_i^2 - v_i^2),$$

after using for this the obvious inequalities

$$\tilde{M}|\zeta_1| - \tilde{M}|\zeta - \zeta_1| \leqslant \tilde{M}|\zeta| \leqslant \tilde{M}|\zeta_1| + \tilde{M}|\zeta - \zeta_1|,$$
$$\tilde{M}|\zeta - \zeta_1| \leqslant \operatorname{Sp} B_1 \left(1 - \frac{\|B_1\|^{1/2}}{1 - \|B_1\|}\right). \qquad (2.9)$$

Finally, estimation $\tilde{M}|\zeta_1|$ is given by the following lemma.

Lemma 4. They occur of the inequality

$$\frac{1}{\sqrt{2\pi}} \sqrt{\operatorname{Sp} B_1} \geqslant \tilde{M}|\zeta_1| \geqslant \frac{1}{\pi} \sqrt{\operatorname{Sp} B_1}. \qquad (2.10)$$

The proof of lemma 4 we will begin from the calculation of the characteristic function a (θ) of random variable ζ_1. The characteristic function of each random variable u_i^2, v_i^2 is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\theta x^2} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{1 - 2i\theta}},$$

so that the function

$$a(\theta) = \prod_i (1 + \rho_i^2 \theta^2)^{-1/2}.$$

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Further, on the basis of the equality

$$\int_{-\infty}^{\infty} \frac{\sin \alpha\theta}{\theta} d\theta = \pi \operatorname{sign} \alpha$$

let us have

$$\mathrm{M}\,|\zeta_1| = \mathrm{M}\,\zeta_1 \operatorname{sign} \zeta_1 =$$
$$= \frac{1}{2\pi i}\,\mathrm{M}\left(\zeta_1 \int_{-\infty}^{\infty} \frac{e^{i\zeta_1\theta} - e^{-i\zeta_1\theta}}{\theta}\,d\theta\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a'(\theta) - a'(-\theta)}{\theta}\,d\theta =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \prod (1 + \rho_i^2\theta^2)^{-1/2} \left(\sum_i \frac{\rho_i^2}{1 + \rho_i^2\theta^2}\right) d\theta.$$

After designating $\frac{\rho_i^2}{\operatorname{Sp} B_1}$ by λ_i, from the last/latter equality we will obtain that

$$\tilde{\mathrm{M}}\,|\zeta_1| = \frac{\sqrt{\operatorname{Sp} B_1}}{2\pi}\int_{-\infty}^{\infty} \prod (1 + \lambda_i\theta^2)^{-1/2} \left(\sum_i \frac{\lambda_i}{1 + \lambda_i\theta^2}\right) d\theta, \quad (2.11)$$
$$\sum_i \lambda_i = 1.$$

Let us find the upper and lower bounds of the functions

$$\Phi(\lambda_1, \ldots, \lambda_n) = \int_{-\infty}^{\infty} \prod (1 + \lambda_i\theta^2)^{-1/2} \left(\sum_i \frac{\lambda_i}{1 + \lambda_i\theta^2}\right) d\theta$$

in simplexes $\sum \lambda_i = 1,\ \lambda_i \geqslant 0.$ The standard method of finding extrema, for example the method of Lagrange's factors, shows with the help of simple calculations that the function Φ reaches the minimum

(maximum) value with equal coordinates $\lambda_1 = \lambda_2 = \ldots$ or in the apex/vertexes of simplex $\sum \lambda_i = 1,\ \lambda_i \geqslant 0$. Consequently, integral in right side (2.11) has as the face side

$$\sup_s \int_{-\infty}^{\infty} \left(1+\frac{\theta^2}{s}\right)^{-\frac{s+2}{2}} d\theta = \sup_s 4\pi \sqrt{s}\,\frac{\Gamma(s-1)}{2^s\left[\Gamma\left(\frac{s}{2}\right)\right]^2} =$$

$$= \lim_{s\to\infty} \sqrt{2\pi}\,\frac{\sqrt{s-2}\,(s-2)^{s-2}\sqrt{s}\,e^2}{s^{s-1}} = \sqrt{2\pi}. \quad (2.12)$$

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Here $\Gamma(s)$ there is Γ Euler's -function, to which we used the Stirling formula. Analogously integral in right side (2.11) is not less

$$\int_{-\infty}^{\infty} (1+\theta^2)^{-3/2}\, d\theta = 2. \quad (2.13)$$

Estimations (2.12) and (2.13) together with equality (2.11) make it possible to write that

$$\frac{1}{\sqrt{2\pi}} \sqrt{\mathrm{Sp}\, B_l} \geqslant \mathsf{M}\,|\zeta_l| \geqslant \frac{1}{\pi} \sqrt{\mathrm{Sp}\, B_l},$$

whereupon from the proof is evident that both estimations are

nonimproved. Lemma 4, and together with it and lemma 3 are demonstrated.

Lemma 5. Undar conditions of lemma 3

$$\int_{-\infty}^{\infty}\left|\frac{p_{12}(x,y)}{p_1(x)\,p_2(y)}-1\right|^2 p_1(x)\,p_2(y)\,dx\,dy=\prod_i(1-\rho_i^2)^{-1}-1. \tag{2.14}$$

This lemma is proven by the direct/straight calculation of left side (2.14).

Let us pass to the proof of theorem 4. Let Q_τ be the measure, generated by our random-process ξ (t) during the association of σ-algebra $\mathfrak{A}(-\infty,-\tau)$ and $\mathfrak{A}(0,\infty)$; let us designate by $\tilde{Q}_\tau$ the measure, which coincides with Q on $\mathfrak{A}(-\infty,-\tau)$ and $\mathfrak{A}(0,\infty)$ and such, that σ-algebra $\mathfrak{A}(-\infty,-\tau)$, $\mathfrak{A}(0,\infty)$ $\tilde{Q}_\tau$ independent. On lemma 2

$$\beta(\tau)=\frac{1}{2}\operatorname{Var}(Q_\tau-\tilde{Q}_\tau). \tag{2.15}$$

Let the process ξ (t) be absolutely regular, i.e., $\lim_{\tau\to\infty}\beta(\tau)=0$. On the basis (2.15) will be located the number τ_0 such that for all $\tau \geqslant \tau_0$

$$\frac{1}{2}\operatorname{Var}(Q_\tau-\tilde{Q}_\tau)=\beta(\tau)<\frac{1}{2}.$$

Consequently, for $\tau \geqslant \tau_0$ and measure Q_τ and $\tilde{Q}_\tau$ are not singular; these measures Gaussian, and on the basis of results §2 o chapter III they are absolutely continuous on each other.

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Let us select in space H $(-\infty,-\tau)$ any base $\eta_{11}, \eta_{12}, \ldots$ in space H $(0, \infty)$ - base $\eta_{21}, \eta_{22}, \ldots$ Let us consider vector $(\eta_{11}, \ldots, \eta_{1n}, \eta_{21}, \ldots, \eta_{2n})$; generated by this vector in the manner that this is shown on page 167, measure Q, $\tilde{Q}$, operator B_1 and the eigenvalues of operator B_1 let us furnish with index n, i.e., to write respectively $Q_n, \tilde{Q}_n, B_{1n}, \rho^2_{in}$. Let us assume $p_n = \frac{dQ_n}{d\tilde{Q}_n}$. To simply count, that

$$M \ln p_n = -\frac{1}{2} \sum \ln(1-\rho^2_{in}). \qquad (2.16)$$

From the absolute continuity of measures $Q_\tau, \tilde{Q}_\tau$ it follows, as this is shown in §2 of chapter III, that

$$\sup_n |M \ln p_n| < \infty. \qquad (2.17)$$

Hence and from (2.16) in turn, escape/ensues the inequality

$$\sup_n \mathrm{Sp}\, B_{1n} < \infty.$$

With $n \to \infty$ the operators B_{1n} weakly converge to operator B_τ, i.e., $\lim_n (B_{1n}\eta, \eta) = (B_\tau\eta, \eta)$ (operators B_{1n} in an obvious manner are determined everywhere in H $(-\infty, \infty)$). Consequently, operator B_τ has the final trace.

Further, let q_τ there is density Q_τ on $\tilde{Q}_\tau$. From results of chapter III, page 108, follow that for $\tau > \tau_0$

$$q_\tau = \tilde{M}(q_{\tau_0} | \mathfrak{A}_\tau),$$

where $\mathfrak{A}_\tau$ is an association of σ-algebra $\mathfrak{A}(-\infty, -\tau)$, $\mathfrak{A}(0, \infty)$. On the strength of (2.14)

$$\tilde{M} | q_\tau - 1 |^2 < \infty.$$

On the basis of the known theorems of the theory of conditional mathematical expectations hence it follows that the random variables q_τ converge in mean quadratic with $\tau \to \infty$ to q_∞. Absolutely regular process ξ (t) is regular, i.e., intersection $\bigcap_\tau \mathfrak{A}(-\infty, -\tau)$ is trivial. Consequently, it is necessary $q_\infty = 1$.

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Actually, random variable $q_\infty = \tilde{M}(q_{\tau_0} | \mathfrak{A}_\infty)$ measured relative to $\mathfrak{A}_\infty = \mathfrak{A}(0, \infty)$ also, for all $A \in \mathfrak{A}_\infty$ regarding the conditional mathematical expectation

$$\int_A q_\infty \, d\tilde{Q} = \int_A q_{\tau_0} \, d\tilde{Q} = \int_A dQ = Q(A) = \tilde{Q}(A) = \int_A 1 \cdot d\tilde{Q}.$$

These equalities mean that $q_\infty = 1$ with probability 1.

On the strength of (2.14)

$$\mathrm{Sp}\, B_\tau \leqslant \tilde{\mathsf{M}}\, |q_\tau - 1|^2 = \tilde{\mathsf{M}}\, |q_\tau - q_\infty|^2 \xrightarrow[\tau\to\infty]{} 0,$$

and the application/use of lemma 3 leads to maximum relationships (2.1).

Conversely, let with a given τ_0 operator B_{τ_0} is a completely continuous operator with the final trace. Then (see observation to theorem 3) process ξ (t) is completely regular and $\|B_\tau\| \to 0$. Let us consider that already $\|B_{\tau_0}\| < 1$. It is obvious also that together with B_{τ_0} all operators B_τ, $\tau > \tau_0$, have the final trace and

$$\mathrm{Sp}\, B_\tau \leqslant \mathrm{Sp}\, B_{\tau_0}.$$

Let us select now, by using the full/total/complete continuity of the operators B_τ, bases $\eta_{11}, \eta_{12}, \ldots, \eta_{21}, \eta_{22}, \ldots$ in the manner that this is shown in lemma 1. Then on the strength of (2.16)

$$\sup_n |\tilde{\mathsf{M}} \ln p_n| \leqslant \mathrm{Sp}\, B_\tau \left(1 + \frac{\|B_\tau\|}{1 - \|B_\tau\|}\right).$$

Being again converted to §2 of chapter III, we see that the measures Q_τ and $\tilde{Q}_\tau$ are absolutely continuous. Furthermore, as already mentioned, process ξ (t) completely was regular and all the more regular. As already it was proved, in that case

$$\tilde{\mathsf{M}}\, |q_\tau - 1|^2 \xrightarrow[\tau\to\infty]{} 0,$$

and, which means, on the strength of (2.14) and $\mathrm{Sp}\, B_\tau \xrightarrow[\tau\to\infty]{} 0$. Now reference to lemma 3 proves theorem.

Observation. In the case of processes with discrete time the theorem allow/assumes the simpler and more final formulation.

In order that the stationary Gaussian process ξ (t) with discrete time would be absolutely regular, it is necessary and sufficient in order that operator B_1 would be completely continuous operator with the final trace.

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Actually, as it was noted in the proof of theorem 3, for processes with discrete time the operators B_τ differ from operator B_1 only by finite-dimensional operator.

Theorem 5. Stationary Gaussian process ξ (t) satisfies the condition of regularity (1.5) in that and only that case, if $B_\tau = 0$ for all $\tau > \tau_0$.

The proof of this simple theorem can be found in monograph [14]. Let us note just the equality $B_\tau = 0$ means that spaces H $(-\infty,-\tau)$, H $(0, \infty)$ are orthogonal and, which means, that σ-algebra $\mathfrak{A}(-\infty, -\tau)$,

$\mathfrak{A}(0, \infty)$ are independent.

Finally, in the following paragraph it will be provel that the conditions of absolute and informational regularity coincide.

§3. Condition of informational regularity.

Theorem 6. Stationary Gaussian process ξ (t) is informationally regular in that and only that case, if for a given τ_0 operator B_{τ_0} is a completely continuous operator with the final trace. In this case and all operators B_τ, $\tau > \tau_0$, are able the final trace, but the informational coefficient of regularity are.

$$I(\tau) = -\frac{1}{2}\sum \ln(1-\rho_i^2) \sim \frac{1}{2}\,\mathrm{Sp}\,B_\tau, \qquad (3.1)$$

where ρ_i^2 is the i eigenvalue of operator B_τ.

We will deduce this theorem from the following common/general/total result. Let $\{\xi(u),\ u \in U;\ \eta(v),\ v \in V\}$ - a given system of Gaussian random variables. Let, further,

$$I_{\xi\eta} = I\{\xi(u),\ u \in U;\ \eta(v),\ v \in V\}$$

- the quantity of information, which is contained in random variables

$\{\xi(u),\ u\in U\}$ relatively $\{\eta(v),\ v\in V\}$. As usual, let H_1, H_2 be the locked linear closures of the corresponding values $\{\xi(u),\ u\in U\}$, $\{\eta(v),\ v\in V\}$; $\mathfrak{A}_1$, $\mathfrak{A}_2$ - σ-algebra, generated by values $\{\xi(u),\ u\in U\}$, $\{\eta(v),\ v\in V\}$; B_1, B_2 - the nonnegative self-adjoint operators, determined by spaces H_1, H_2 since this is shown above. Then is valid the following theorem [1].

FOOTNOTE [1]. I. M. Gelfand, A. M. Yaglom. On the calculation of quantity of information about the random function, which is contained in another such function, UMN XII, V. 1 (1957). ENDFOOTNOTE.

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Theorem 7. Quantities of information $I_{\xi\eta}$ certainly in that and only that case, if operator B_1 (but that means and B_2) is a completely continuous operator with the final trace, but $\|B_1\| < 1$. In this case

$$I_{\xi\eta}=-\frac{1}{2}\sum_i \ln(1-\rho_i^2), \tag{3.2}$$

where $\rho^2_1 \geqslant \rho^2_2 \geqslant \ldots$ - all eigenvalues of operator B_1.

Proof of theorem 7.1. Let us count first the quantity of

information, which is contained in one Gaussian random variable ξ with respect to another, Gaussian to the connected with it value η. It is understandable that it is possible to count

$$M\xi = M\eta = 0, \quad M\xi^2 = M\eta^2 = 1.$$

Let, furthermore, $M\xi\eta = \rho$, so that the combined density of our values is equal to

$$p(x, y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)}\right\}.$$

We have

$$I_{\xi\eta} = \sup_{A_i, B_j} \sum \ln \frac{P\{A_i B_j\}}{P\{A_i\} P\{B_j\}} P\{A_i B_j\}, \qquad (3.3)$$

where the face side it is taken on the events of form $A_i = \{\xi \in E_i\}$, $B_j = \{\eta \in E'_j\}$, where E_i, E'_j are linear borel sets. Consequently,

$$I_{\xi\eta} = \sup_{E_i, E'_j} \sum \ln \frac{Q_{12}\{E_i \times E'_j\}}{Q_1\{E_i\} Q_2\{E'_j\}} Q_{12}(E_i \times E'_j), \qquad (3.4)$$

where Q_{12}, Q_1, Q_2 are measures on plane and straight line, induced respectively with random number distributions (ξ, η) and ξ, η:

$$Q_{12}(E \times F) = P\{\xi \in E, \ \eta \in F\},$$
$$Q_1(E) = P\{\xi \in E\}, \quad Q_2(F) = P\{\eta \in F\}.$$

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Let first $\rho < 1$. Then measure Q_{12} is absolutely continuous with Lebesgue, measure Q_{12}, $Q_1 \times Q_2$ are mutually absolutely continuous

with the density

$$\frac{Q_{12}(dx\,dy)}{Q_1(dx)\,Q_2(dy)} = \frac{p(x,y)}{p(x)\,p(y)},$$

where $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ - the density of distribution of values ξ, η. Therefore the sum of right side (3.4) is an integral sum for Lebesgue's integral function ln p (x, y)/p (x) p (y) p (x, y). Consequently,

$$I_{\xi\eta} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ln \frac{p(x,y)}{p(x)\,p(y)}\, p(x,y)\,dx\,dy = \\ = \mathsf{M} \ln \frac{p(\xi,\eta)}{p(\xi)\,p(\eta)} = -\frac{1}{2}\ln(1-\rho^2). \qquad (3.5)$$

If ρ = 1, then $I_{\xi\eta} = \infty$. In order to be convinced of this, is sufficient to select in (3.3) $A_1 = B_1, \ldots, A_n = B_n$ with probabilities $\mathsf{P}(A_i) = \frac{1}{n}$. We will obtain that with all n

$$I_{\xi\eta} > \ln n.$$

Since with ρ = 1 right side (3.5) also goes to infinity, it is possible to consiier formula (3.5) valid, also, in this case.

2. Let now sets U, V be final. Quantity of information $I_{\xi\eta}$ is invariable relative to the nondegenerate transforms of spaces H_1, H_2. This immediately follows from the determination of the quantity of

information (1.6) and of that obvious fact that σ-algebra $\mathfrak{A}_1, \mathfrak{A}_2$ are invariant relativa to the transforms indicated.

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Therefore, and also on the strength of lemma 1, we can from the very beginning count

$$\{\xi(u),\ u\in U\}=(\xi_1,\ \dots,\ \xi_n),\quad \{\eta(v),\ v\in V\}=(\eta_1,\ \dots,\ \eta_n),$$

where all vapors (ξ_i, η_i) were independent and

$$M\xi_i=M\eta_i=0,\quad D\xi_i=D\eta_i=1,\quad M\xi_i\eta_i=\rho_i.$$

In (1.6) is sufficient to take sup on events A_i, B_j:

$$A_i=\{\xi_1\in E_{i1},\ \dots,\ \xi_n\in E_{in}\},\ B_j=\{\eta_1\in E'_{j1},\ \dots,\ \eta_n\in E'_{jn}\}.$$

From independence the vapor (ξ_i, η_i) follows then that

$$I_{\xi\eta}=\sum_{i=1}^{n} I_{\xi_i\eta_i},$$

and, which means, on the strength of (3.5)

$$I_{\xi\eta}=-\frac{1}{2}\sum_{i=1}^{n}\ln(1-\rho_i^2),$$

whereupon $I_{\xi\eta}<\infty$ in that and only that case, if $\rho_1=\sqrt{\|B_1\|}<1$.

3. Let us pass to the examination of the general case. Are

above, in p. 1 proofs, has already been noted that if $\|B_1\|=1$, then $I_{\xi\eta}=\infty$. Therefore it suffices to consider the case $\|B_1\|<\infty$.

Let B_1 be operator with the final trace. Let us select any n of random variables $\xi_1, \ldots, \xi_n$ from family $\{\xi(u), u\in U\}$ and n of random variables $\eta_1, \ldots, \eta_n$ from family $\{\eta(v), v\in V\}$. By $B_1^{(n)}$ let us designate the operator, which corresponds to this selection of random variables. It goes without saying that operators $B_1^{(n)}$ can be examined in all space H_{12}, stretched on $\{\xi(u), \eta(v)\}$. Let us designate $\lambda_1^{(n)}\geqslant\lambda_2^{(n)}\geqslant\ldots\geqslant\lambda_n^{(n)}$ the eigenvalues of operator $B_1^{(n)}$ (it is finite). It is obvious, $\lambda_1^{(n)}=\|B_1^{(n)}\|\leqslant\|B_1\|<1$.

Random variables $(\xi_1, \ldots, \xi_n), (\eta_1, \ldots, \eta_n)$ can be chosen so that the operators $B_1^{(n)}$ weakly converge to B_1. Then

$$\lim_n \mathrm{Sp}\, B_1^{(n)}=\lim_n\sum_i \lambda_i^{(n)}=\infty.$$

From the determination of quantity of information (1.6) it follows that

$$I_{\xi\eta}\geqslant I((\xi_1, \ldots, \xi_n), (\eta_1, \ldots, \eta_n)).$$

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At the same time as it is already establish/installed in p. 2,

$$I((\xi_1, \ldots, \xi_n), (\eta_1, \ldots, \eta_n)) = -\frac{1}{2}\sum_i \ln(1-\lambda_i^{(n)}) =$$
$$= \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\sum_i (\lambda_i^{(n)})^k \geqslant \frac{1}{2}\,\mathrm{Sp}\,B_1^{(n)}.$$

Consequently, if $SpB_1 = \infty$, then also $I_{\xi\eta} = \infty$.

Let now B_1 there is a completely continuous operator with the final trace. On the basis of lemma 1 let us select of spaces H_1, H_2 bases $(\xi_1, \xi_2, \ldots)$, $(\eta_1, \eta_2, \ldots)$, that consist of the personal cell/elements of the operators B_1, B_2. During this selection $M\xi_i\eta_i = \rho_i$, where ρ_i^2 are the eigenvalues of operators B_1, B_2.

Any events $A \in \mathfrak{A}_1$, $B \in \mathfrak{A}_2$ it is possible, by choosing n by sufficient large as conveniently to approach well by the events, measurable relative to random variables $(\xi_1, \ldots, \xi_n)$, $(\eta_1, \ldots, \eta_n)$. Being again converted to the determination of quantity of information (1.6), is concluded hence, that

$$I_{\xi\eta} = \lim_{n\to\infty} I(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n),$$

where

$$I(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n) = -\frac{1}{2}\sum_1^n \ln(1-\rho_i^2).$$

Theorem 7 is demonstrated.

From this theorem immediately it follows that the random process $\xi(t)$ is informationally regular only if the operators B_τ have the final trace for $\tau > \tau_0$, and also that occurs equality (3.1). In the proof of theorem 4 it was establish/installed that $\lim_\tau \mathrm{Sp}\, B_\tau = 0$, if only $\mathrm{Sp}\, B_{\tau_0} < \infty$ for any τ_0. Theorem 6 is also demonstrated.

Equate/comparing theorems 4, 6, we are convinced that the conditions of absolute and informational regularity are equivalent and that, for example, with $I(\tau) \to 0$

$$\frac{1}{\pi^2} \leqslant \overline{\lim} \frac{\beta^2(\tau)}{I(\tau)} \leqslant \overline{\lim} \frac{\beta^2(\tau)}{I(\tau)} \leqslant \frac{1}{2\pi}.$$

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Observation. On the basis of observation to theorem 4 stationary Gaussian process $\xi(t)$ with discrete time is informationally regular in that and only that case, if operator B_1 has the final trace.

§4. Condition of absolute regularity. Processes with discrete time.

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GAUSSIAN RANDOM PROCESSES. PART 1,(U)
NOV 77 I A IBRAGIMOV, Y A ROZANOV

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In this and following paragraph we will give the full/total/complete and sufficiently efficient description of absolutely regular stationary Gaussian processes (processes, which satisfy condition (1.3)). In accordance with results §3 simultaneously this will give description of informationally regular processes.

Theorem 8 [1].

FOOTNOTE [1]. I. A. Ibragimov, V. N. Solev, Obodn the condition of the regularity of the stationary Gaussian process, PAS of the USSR, 185, No 3, 509-512. ENDFOOTNOTE.

Stationary Gaussian process $\xi(t)$ with the discrete time $t = 0, \pm 1, \ldots$ is absolutely regular (but that means and informationally is regular) in that and only in such a case, when it has spectral density $f(\lambda)$, representable in the form

$$f(\lambda) = |P(e^{i\lambda})|^2 a(\lambda), \qquad (4.1)$$

where $P(z)$ - polynomial with roots on circumference $|z| = 1$, and

coefficients a_j Fourier series $\sum a_j e^{i\lambda j}$ function ln a (λ) are such, that

$$\sum |j| |a_j|^2 < \infty. \qquad (4.2)$$

Proof is based on the use of theorem 4, more precise, observation to it. It goes without saying that the analytical formulation of theorem 8 requires passage of space H (-∞, ∞) to isometric function space L (F). In order not to introduce excess designations, operators into L (F), analogous to the operators $B_\tau^\pm$, also let us designate through $B_\tau^\pm$.

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Any absolutely regular process is regular and, consequently (see §2 of chapter II), has spectral density f (λ), that allow/assumes representation of the form

$$f(\lambda) = |g(e^{i\lambda})|^2, \qquad (4.3)$$

where g (z) - the external function of Hardy's class $\mathcal{H}^2$ in circle.

Let us introduce into examination function $c(\lambda) = \frac{g(e^{i\lambda})}{\overline{g(e^{i\lambda})}}$. By $\sum_{j=-\infty}^{\infty} c_j e^{ij\lambda}$ let us designate its Fourier series.

Lemma 6. Process ξ (t) is absolutely regular in that and only that case, if descends the series

$$\sum_{-\infty}^{0} |j| |c_j|^2. \qquad (4.4)$$

In accordance with observation to theorem 4 (see page 182) it is necessary to demonstrate that the convergence of series (4.4) is equivalent to operator's nuclearity B_1. To us will somewhat more conveniently deal with unitary equivalent to it operator $B_1^+ = \mathcal{P}_0^+ \mathcal{P}_1^- \mathcal{P}_0^+$, where this time $\mathcal{P}_\tau^+$ - projector in L (F) on $L_{(\tau, \infty)}(F)$, $\mathcal{P}_\tau^-$ - projector in L (F) on $L_{(-\infty, -\tau)}(F)$. Is sufficient to consider B_1^+ as operator from $L_{(0, \infty)}(F) = L^+(F)$ in $L^+(F)$. As is known [1], operator B_1^+ will be nuclear (i.e. will have the final trace) in that and only that case, if for a given orthonormalized base $\{e_j\}$ in $L^+(F)$ is descend a series $\sum_j (B_1^+ e_j, e_j)_F$.

FOOTNOTE [1]. See, for example, [5], page 55; recall that all operators B_τ^+ are positive. ENDFOOTNOTE.

If this series descends for any the orthonormalized base, it descends for any such base, and its sum and is equal to operator's trace B_1^+.

Let us select as the orthonormalized base in $L^+(F)$ of function $e_k(\lambda) = e^{i\lambda k} g^{-1}(e^{i\lambda})$, $k = 0, 1, \ldots$ The orthonomality of system $\{e_k\}$ is obvious;

let us demonstrate that this system is a base.

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Actually, in §2 of chapter II is shown, that any cell/element $\varphi \in L^+(F)$ can be registered in the form ϕ_1/g, where $\varphi_1 \in \mathscr{H}^2$. If $\sum c_k e_k$ is any linear combination of cell/elements e_k, then [1]

$$\|\varphi - \sum c_k e_k\|_F = \|\varphi_1 - \sum c_k e^{i\lambda k}\|^{(2)}.$$

FOOTNOTE [1]. Recall that $\|\varphi\|^{(p)}$ indicates norm in space $\mathscr{L}^p(-\pi, \pi)$. ENDFOOTNOTE.

Since the linear combinations $\sum c_k e^{i\lambda k}$ are dense in $\mathscr{H}^2$, the last/latter equality means that the linear combinations $\sum c_k e_k$ are dense in $L^+(F)$, i.e., that $\{e_k\}$ there is a base in $L^+(F)$.

Analogous with lemma 2 chapters II are proven the following equalities:

$$\mathscr{P}_\tau^+ = g^{-1}\Pi_\tau^+ g, \quad \mathscr{P}_\tau^- = \bar{g}^{-1}\Pi_\tau^- \bar{g},$$

where $\Pi_\tau^+ (\Pi_\tau^-)$ essence projectors in $\mathscr{L}^2(-\pi, \pi)$ on $e^{i\lambda\tau}\mathscr{H}^2$ (on Therefore $e^{-i\lambda\tau}\mathscr{H}^{2-}$).

$$\operatorname{Sp} B_1^+ = \sum_{k=0}^{\infty} (B_1^+ e_k, e_k)_F = \sum_{k=0}^{\infty} (\mathscr{P}_0^+ \mathscr{P}_1^- \mathscr{P}_0^+ e_k, e_k)_F =$$
$$= \sum_{k=0}^{\infty} (\mathscr{P}_1^- e_k, e_k)_F = \sum_{k=0}^{\infty} \left(\Pi_1^- \frac{\bar{g}}{g} e^{i\lambda k}, \frac{\bar{g}}{g} e^{i\lambda k} \right)^{(2)} =$$
$$= \sum_{k=0}^{\infty} \left(\left\| \Pi_1^- \frac{\bar{g}}{g} e^{i\lambda k} \right\|^{(2)} \right)^2 = \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} \left| \sum_{j=-\infty}^{-1} \bar{c}_{j-k} e^{i\lambda j} \right|^2 d\lambda =$$
$$= \sum_{j=-\infty}^{0} |c_j|^2 |j|. \qquad (4.5)$$

Lemma is demonstrated.

To us it remained to demonstrate that the conditions of theorem 8 were equivalent to the conditions of lemma 6.

Is checked that satisfaction of the conditions of theorem will draw satisfaction of the conditions of lemma 6. From (4.2) it follows that $\ln a \in \mathscr{L}^2(-\pi, \pi)$, that is more $\ln a \in \mathscr{L}^1(-\pi, \pi)$. Therefore function a (λ) can be registered in the form $a(\lambda) = |g_1(e^{i\lambda})|^2$, where g_1 is an external function of class $\mathscr{H}^2$. Consequently, and $f(\lambda) = |g(e^{i\lambda})|^2$, where the external function g (z) = P (z) g_1 (z).

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If $e^{i\lambda_1}, \ldots, e^{i\lambda_\nu}$ - zero polynomial P (z), then, obviously,

$$\frac{\overline{P(e^{i\lambda})}}{P(e^{i\lambda})} = e^{i\alpha} \prod_{s=1}^{\nu} \frac{e^{-i\lambda} - e^{-i\lambda_s}}{e^{i\lambda} - e^{i\lambda_s}} = e^{i\alpha} e^{-i\nu\lambda} (-1)^{\nu},$$

where a is real number. Therefore it suffices to trace function $c_1(\lambda) = \frac{\overline{g_1(e^{i\lambda})}}{g_1(e^{i\lambda})} = \sum c_{1j} e^{i\lambda j}$ and to demonstrate that descends a series $\sum\limits_{j<0} |c_{1j}|^2 |j|$. We will demonstrate even convergence of series $\sum\limits_j |j| |c_{1j}|^2$. For this by us will be required the following.

Lemma 7. Let the function $h(\lambda) \sim \sum h_j e^{i\lambda j}$ belong $\mathscr{L}^2(-\pi, \pi)$. Let us designate by ω (δ; h) the module/modulus of the continuity of function h in metric $\mathscr{L}^2$, i.e.,

$$\omega(\delta;\, h) = \sup_{|\theta| \leqslant \delta} \left(\int_{-\pi}^{\pi} |h(\lambda+\theta) - h(\lambda)|^2\, d\lambda \right)^{1/2}.$$

Then the inequalities

$$\sum_1^{\infty} \omega^2\left(\frac{1}{n};\, h\right) < \infty, \tag{4.6}$$

$$\sum |j| |a_j|^2 < \infty \tag{4.7}$$

are equivalent.

Actually,

$$\sum_n \omega^2\left(\frac{1}{n};\, h\right) = \sum_{n=1}^{\infty} \sup_{\delta \leqslant \frac{1}{n}} 4 \sum_{j=-\infty}^{\infty} |h_j|^2 \sin^2 \frac{j\delta}{2} \leqslant$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{|j| \leqslant n} |h_j|^2 j^2 + 4 \sum_{n=1}^{\infty} \sum_{|j|>n} |h_j|^2 =$$

$$= \sum_{j=-\infty}^{\infty} |h_j|^2 j^2 \sum_{n>|j|} \frac{1}{n^2} + 4 \sum_{j=-\infty}^{\infty} |j| |h_j|^2 \leqslant 5 \sum_j |j| |h_j|^2.$$

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On the contrary, for $|x|<1$ $\frac{\sin x}{x}>1-\frac{x^2}{6}$, and therefore

$$\sum_n \omega^2\left(\frac{1}{n};\, h\right) \geqslant 4\sum_{n=1}^{\infty}\sum_{|j|\leqslant n}|h_j|^2\sin^2\frac{j}{2n}\geqslant$$
$$\geqslant \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{|j|\leqslant n}|h_j|^2 j^2=$$
$$=\frac{1}{2}\sum_{j=-\infty}^{\infty}|h_j|^2 j^2\sum_{n>|j|}\frac{1}{n^2}>\frac{1}{4}\sum|j||h_j|^2.$$

Lemma is demonstrated.

Being returned to the proof of theorem, let us note, that $g_1=\exp\left\{\frac{1}{2}(\ln a+i\widetilde{\ln}\, a)\right\}$ and that, therefore, $\bar{g}_1/g_1=\exp\{-i\widetilde{\ln}\, a\}$, where, generally, through $\tilde{h}$ is designated function, (harmonically) conjugated/combined h. The absolute values of the module/moduli of the Fourier coefficients of functions ln a and $\widetilde{\ln}$ a coincide, and therefore ω (δ; ln a) = ω (δ; $\widetilde{\ln}$ a). Further,

$$|\exp\{-i\widetilde{\ln}\, a(\lambda+\delta)\}-\exp\{-i\widetilde{\ln}\, a(\lambda)\}|=$$
$$=|\exp\{-i(\widetilde{\ln}\, a(\lambda+\delta)-\widetilde{\ln}\, a(\lambda))\}-1|\leqslant$$
$$\leqslant|\widetilde{\ln}\, a(\lambda+\delta)-\widetilde{\ln}\, a(\lambda)|.$$

Therefore and

$$\omega\left(\delta;\ \frac{\tilde{g}_1}{g_1}\right)\leqslant\omega(\delta;\ \widetilde{\ln}\, a)=\omega(\delta;\ \ln a).$$

Being converted to lemma 7, we see that the convergence of series $\sum|j||a_j|^2$ will draw convergence of series $\sum_j|j||c_{1j}|^2$. In summation, we demonstrated that conditions (4.1), (4.2) theorem 8 were sufficient for the absolute regularity of Gaussian process ξ (t).

The need for these conditions is proven more complexly.

Let the process ξ (t) be absolutely regular. Then it is regular and has spectral density f (λ), whereupon $\ln f\in\mathscr{L}^{f}(-\pi,\pi)$.

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Let us introduce into examination the coefficient (full/total/complete) of the regularity

$$\rho(\tau;\ f)=\sup_{\varphi,\ \psi}\left|\int_{-\pi}^{\pi}\varphi(e^{i\lambda})\overline{\psi(e^{i\lambda})}f(\lambda)\,d\lambda\right|=\sup|(\varphi,\ \psi)_F|,$$

where sup it is taken on all ϕ, ψ, that belongs with respect to the single spheres of spaces $L_{(0,\infty)}(F)$, $L_{(-\infty,-\tau)}(F)$.

In §1 has already been noted that any absolutely regular Gaussian process satisfies the condition of powerful mixing and, which means, completely regular, i.e., $\lim\limits_{\tau\to\infty} \rho(\tau) = 0$. Specifically, will be located such a k that $\rho(k)<1$. According to theorem 4 chapters V spectral density f (λ) it is possible to register in the form

$$f(\lambda)=|P(e^{i\lambda})|^2 a(\lambda),$$

where P (z) - polynomial with roots on $|z|=1$, and function $a=e^{u+\tilde{v}}$, $\|u\|^{(\infty)}<\infty$, $\|v\|^{(\infty)}<\pi/2$. On the basis of the same theorem the coefficients of regularity ρ (τ; a) and ρ (τ; $1/a$), constructed according to the spectral densities a and $1/a$, are such, that

$$\rho(1;\ a)<1, \quad \rho\left(1;\ \frac{1}{a}\right)=\rho<1. \qquad (4.8)$$

Obviously, function a (λ) can be registered as $|g_1(e^{i\lambda})|^2$, where g_1 is an external function from $\mathcal{H}^2$. If we, as in the proof of sufficiency, place

$$c_1(\lambda)=\frac{\overline{g_1(e^{i\lambda})}}{g_1(e^{i\lambda})}=\sum_l c_{1l}e^{i\lambda l},$$

that of lemma 6, relying on the already used above equality $\frac{\overline{P(e^{i\lambda})}}{P(e^{i\lambda})} = e^{i\alpha}e^{-i\nu\lambda}(-1)^{\nu}$, it is easy to deduce convergence of series $\sum_{-\infty}^{\infty} |j||c_{1j}|^2$.

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If this series descends, then

$$\sum_{n=0}^{\infty} \inf_{A \in H^2} (\| c_1(\lambda) - e^{-in\lambda} A(\lambda) \|^{(2)})^2 \leqslant$$
$$\leqslant \sum_{n=0}^{\infty} \sum_{-\infty}^{-(n+1)} |c_{1j}|^2 = \sum_{-\infty}^{0} |j||c_{1j}|^2 < \infty. \qquad (4.9)$$

Let us select the sequence of polynomials $A_n(z)$ so that

$$\sum_{n=0}^{\infty} (\| v_1(\lambda) - e^{-in\lambda} A_n(e^{i\lambda}) \|^{(2)})^2 < \infty, \qquad (4.10)$$

let us assume $e^{-in\lambda} g_1 A_n = Q_n(e^{-i\lambda}) + B_n$, where $Q_n(z)$ - the polynomial of degree is not higher than n, a $B_n \in \mathscr{H}^2$. On the strength of (4.8)

$$
\begin{aligned}
&(\| c_1(\lambda) - e^{-in\lambda} A_n(\lambda) \|^{(2)})^2 = \\
&\quad = \int_{-\pi}^{\pi} |(\overline{g_1(e^{i\lambda})} - Q_n(e^{-i\lambda})) - B_n(e^{i\lambda})|^2 \frac{d\lambda}{a(\lambda)} \geqslant \\
&\geqslant \int_{-\pi}^{\pi} |\overline{g_1(e^{i\lambda})} - Q_n(e^{-i\lambda})|^2 \frac{d\lambda}{a(\lambda)} + \int_{-\pi}^{\pi} |B_n(e^{i\lambda})|^2 \frac{d\lambda}{a(\lambda)} - \\
&\quad - 2 \left| \int_{-\pi}^{\pi} (\overline{g_1(e^{i\lambda})} - Q_n(e^{-i\lambda}))\, \overline{B_n(e^{i\lambda})} \frac{d\lambda}{a(\lambda)} \right| \geqslant \\
&\geqslant \int_{-\pi}^{\pi} |g_1(e^{i\lambda}) - Q_n(e^{-i\lambda})|^2 \frac{d\lambda}{a(\lambda)} + \int_{-\pi}^{\pi} |B_n(e^{i\lambda})|^2 \frac{d\lambda}{a(\lambda)} - \\
&- 2\rho \left(\int_{-\pi}^{\pi} |\overline{g_1(e^{i\lambda})} - Q_n(e^{-i\lambda})|^2 \frac{d\lambda}{a(\lambda)} \int_{-\pi}^{\pi} |B_n(e^{i\lambda})|^2 \frac{d\lambda}{a(\lambda)} \right)^{1/2} \geqslant \\
&\qquad \geqslant (1-\rho) \int_{-\pi}^{\pi} |\overline{g_1(e^{i\lambda})} - Q_n(e^{-i\lambda})|^2 \, d\lambda.
\end{aligned}
$$

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Therefore together with together (4.10) descends the series

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \inf_{P_n} \int_{-\pi}^{\pi} |g_1(e^{i\lambda}) - P_n(e^{i\lambda})|^2 \frac{d\lambda}{a(\lambda)} = \\
&\qquad = \sum_{n=0}^{\infty} \inf_{P_n} \int_{-\pi}^{\pi} |g_1(e^{i\lambda}) - P_n(e^{i\lambda})|^2 \frac{d\lambda}{a(\lambda)}, \qquad (4.11)
\end{aligned}
$$

where inf it is taken on all polynomials $P_n(e^{i\lambda})$ **degree not higher**

than n.

Our problem was reduced to research of the properties of the best approach/approximations of function g_1 (λ) by polynomials in space metrics $\mathscr{L}^2$ with weight $1/a$. It is logical for this purpose to introduce the polynomials $\varphi_\nu(z;\,1/a)=\varphi_\nu(z),\ \nu=0,\,1,\,\ldots,$ orthogonal with weight $1/a$. For reader's convenience a series of the properties of such polynomials is formulated below in the form of several lemmas. The proof of these lemmas can be found in many management/manuals on orthogonal polynomials; we, in particular, everywhere refer below to monograph [11] of U. Grenander and G. Sege, the most close to the specialists in the probability theory.

Let w (λ) - the nonnegative summarized function on $[-\pi,\,\pi]$. Let us additionally assume that $\ln w\in\mathscr{L}^1(-\pi,\,\pi)$, so that $w(\lambda)=|\gamma(e^{i\lambda})|^2$, where γ (z) - the external function of class $\mathscr{H}^2$. Orthogonal with weight w polynomials $\varphi_0(z;\,w),\ \varphi_1(z;\,w),\ \ldots,\ \varphi_\nu(z;\,w),\ \ldots$ are called polynomials such, what

a) φ_ν - polynomial of degree ν with the positive coefficient with leading term;

b) $$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\varphi_\nu(z;\,w)\,\overline{\varphi_\mu(z;\,w)}\,w(\lambda)\,d\lambda=\delta_{\nu\mu},\qquad z=e^{i\lambda}.$$

is placed, further, $\varphi_\nu^*(z;\ w) = z^n \bar{\varphi}_\nu(z^{-1})$. Polynomials $\varphi_\nu, \varphi_\nu^*$ are connected by following relationship/ratios (see [11], by page 58):

$$k_n z\varphi_n(z) = k_{n+1}\varphi_{n+1}(z) - l_{n+1}\varphi_{n+1}^*(z),$$
$$k_n\varphi_{n+1}(z) = k_{n+1}z\varphi_n(z) + l_{n+1}\varphi_n^*(z), \qquad (4.12)$$

where through k_n, l_n they are designated respectively the coefficient with z^n and the absolute term of polynomial φ_n.

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Let us designate by c_ν ν -1 Fourier coefficients functions w and is placed

$$D_n(w) = \det \| c_{\nu-\mu} \|, \qquad \nu, \mu = 0, 1, \ldots, n.$$

Determinants $D_n(w)$ are called concentrating definitions, which correspond to weight (function) w. They occur of equality ([11], of page 54)

$$k_n(w) = \left(\frac{D_{n-1}(w)}{D_n(w)}\right)^{1/2}, \qquad n = 1, \ldots \qquad (4.13)$$

Let us determine, further, geometrical mean G (w) of function w by the equality

$$G(w) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln w(\lambda)\, d\lambda\right\}.$$

Lemma 9. Let function w (λ) be determined and summarized on $[-\pi, \pi]$, w $\geqslant$ 0 and $\ln w \in \mathscr{L}^{(1)}$. Let as before $w(\lambda) = |\gamma(e^{i\lambda})|^2$, where γ

are an external function from $\mathscr{H}^2$. Let finally $\varphi_\nu(z) = \varphi_\nu(z;\ w)$ be the orthogonal polynomials, associated with weight w. Then

$$s(z;\ w) = s(z) = \sum_{\nu=0}^{\infty} \overline{\varphi_\nu(0)}\, \varphi_\nu(z) = \frac{1}{\gamma(0)}\, \frac{1}{\gamma(z)},\quad |z| < 1, \tag{4.14}$$

$$\lim_{n\to\infty} k_n(w) = [G(w)]^{-1/2} = \frac{1}{\gamma(0)}, \tag{4.15}$$

it is evenly in any circle $|z| \leqslant r < 1$

$$\lim_{n\to\infty} \varphi_n^*(z) = \gamma(z). \tag{4.16}$$

Lemma 9. Under conditions of lemma 8

$$k_n^2(w) = \frac{D_{n-1}(w)}{D_n(w)} = \sum_0^n |\varphi_\nu(0;\ w)|^2,$$

$$G(w) = \left(\sum_0^\infty |\varphi_\nu(0,\ w)|^2\right)^{-2}. \tag{4.17}$$

The proof of first equality (4.17) see in [11], page 56; second equality (4.17) follows from the first and formula (4.15).

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Lemma 10. If $w(\lambda) = \frac{1}{|\Gamma(e^{i\lambda})|^2}$, where $\Gamma(z)$ is a polynomial of degree p, everything zero which they lie/rest at circle $|z| < 1$, then for $\nu \geqslant p$

$$\varphi_\nu(z) = z^{\nu-p}\Gamma(z).$$

Proof. If $\nu \geqslant p$, a $\mu < \nu$, then

$$\int_{-\pi}^{\pi} e^{-i\mu\lambda}\varphi_\nu(e^{i\lambda})\,w(\lambda)\,d\lambda = \frac{1}{i}\int_{|z|=1}\frac{z^{\nu-\mu-1}}{z^p\overline{\Gamma(z)}} = 0, \quad (4.18)$$

then since integrand is analytical in circle $|z| \leqslant 1$. Actually, if $\Gamma(z) = \sum_0^p \gamma_j z^j$, then on $|z| = 1$

$$z^p\overline{\Gamma(z)} = \sum_0^p \bar{\gamma}_j z^{p-j},$$

and, therefore, function $(z^p\overline{\Gamma(z)})^{-1}$ allow/assumes analytical continuation in circle $|z| < 1$. Equality (4.18) means that the polynomials $\varphi_\nu(e^{i\lambda})$ are orthogonal to all functions $e^{i\mu\lambda}$, $\mu < \nu$. The equality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi_\nu(e^{i\lambda})|^2\,w(\lambda)\,d\lambda = 1$$

is obvious. Lemma is demonstrated.

Lemma 11. Under conditions of lemma 10 for all $\nu \geqslant p$

$$\frac{D_\nu(w)}{[G(w)]^{\nu+1}} = \frac{D_p(w)}{[G(w)]^{p+1}} = \exp\left\{\frac{1}{\pi}\iint_{|z|\leqslant 1}\left|\frac{\gamma'(z)}{\gamma(z)}\right|^2 d\sigma\right\}, \quad (4.19)$$

where as before $\gamma(z)$ the external function of class $\mathscr{H}^2$, determined

by equality $w = |\gamma|^2$. Integration is conducted to the right according to circle $|z| \leq 1$.

Proof see in [11], page 102.

end section.

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Now we have all the necessary information relative to orthogonal polynomials and can return to the proof of our fundamental theorem 8.

Thus, let $\varphi_\nu(z)=\varphi_\nu(z;\ 1/a)$ – the orthogonal polynomials, which correspond to weight 1/a. Let us note that $\frac{1}{|a(\lambda)|^2}=|\gamma(e^{i\lambda})|^2$, where the external function $\gamma(e^{i\lambda})=\frac{1}{g_1(e^{i\lambda})}$. Hence on the basis of formula (4.14) it follows that Fourier coefficients function $g_1 = 1/\gamma$ in orthogonal system $\{\varphi_\nu(e^{i\lambda})\}$ essence $\overline{\varphi_\nu(0)}\,\overline{\gamma(0)}$. Therefore

$$\inf_{P_n}\int_{-\pi}^{\pi}|g_1(e^{i\lambda})-P_n(e^{i\lambda})|^2\frac{d\lambda}{a(\lambda)}=|\gamma(0)|^2\sum_{n+1}^{\infty}|\varphi_\nu(0)|^2.$$

Convergence of series (4.11) makes it possible to claim that also

$$\sum_{n=0}^{\infty}\sum_{\nu=n+1}^{\infty}|\varphi_\nu(0)|^2=\sum_\nu \nu|\varphi_\nu(0)|^2<\infty. \qquad (4.20)$$

Further our target/purpose is to show that the convergence of series (4.20) will draw convergence of series (4.2). For this purpose besides polynomials $\varphi_n(z)$ let us examine polynomials $\varphi_{\nu n}(z)=\varphi_n(z;\ |\varphi_n|^{-2})$ – the orthogonal polynomials, associated with weight $|\varphi_n(e^{i\lambda})|^{-2}$. As is known ([11], page 57), zero orthogonal polynomials $\varphi_n(z;\ w)$, which

correspond to any weight w, they lie/rest at region $|z| < 1$. Consequently, zero polynomials $\varphi_n^*(z) = z^n(\bar{\varphi}_n(z^{-1})$ they lie/rest outside circle $|z| \leq 1$, so that $\varphi_n^*(z)$ – external functions from $\mathcal{H}^2$. Furthermore, on $|z| = 1$ occurs equality $|\varphi_n|^2 = |\varphi_n^*|^2$, so that $(\varphi_n^*)^{-1}$ – the external function, which corresponds to weight $|\varphi_n|^{-2}$.

Further, on lemma 10

$$\varphi_{\nu n}(z) = z^{n-\nu}\varphi_n(z), \qquad \nu \geqslant n.$$

Specifically, $\varphi_{nn}(z) = \varphi_n(z)$. From this equality and identities (4.12) it follows that

$$\varphi_{\nu n}(z) = \varphi_\nu(z), \qquad \nu \leqslant n. \tag{4.21}$$

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Orthogonal polynomials $\varphi_\nu(z)$, it goes without saying, are linearly independent, so that with all k of function $e^{ik\lambda}$ essence the linear combinations of polynomials $\varphi_\nu(z)$, $0 \leqslant \nu \leqslant k$. Hence and from (4.21) it follows that first n + 1 Fourier coefficients $c_0, \ldots, c_n$ (but that means and $c_{-n}, \ldots, c_{-1}, c_0$) the functions of $\frac{1}{a(\lambda)}$ and $\frac{1}{|\varphi_n(e^{i\lambda})|^{-2}}$ they coincide. Consequently,

$$D_s(|\varphi_n|^{-2}) = D_s\left(\frac{1}{a}\right), \qquad s \leqslant n.$$

By using lemmas 8, 9 let us find with the help of the last/latter equality that for all s $\geqslant$ n

$$\ln \frac{D_s(|\varphi_n|^{-2})}{G(|\varphi_n|^{-2})^{s+1}} = \ln \frac{D_n(|\varphi_n|^{-2})}{G(|\varphi_n|^{-2})^{n+1}} =$$
$$= \sum_{1}^{n} \ln \frac{D_k(|\varphi_n|^{-2})}{D_{k-1}(|\varphi_n|^{-2})} + \ln D_0(|\varphi_n|^{-2}) - (n+1)\ln G(|\varphi_n|^{-2}) =$$
$$= \sum_{k=0}^{n} \ln \frac{\sum_{0}^{n} |\varphi_\nu(0)|^2}{\sum_{0}^{k} |\varphi_\nu(0)|^2} = -\sum_{k=0}^{n-1} \ln \left(1 - \frac{\sum_{k+1}^{n} |\varphi_\nu(0)|^2}{\sum_{0}^{n} |\varphi_\nu(0)|^2} \right). \quad (4.22)$$

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Further, set/assuming $\frac{\sum_{k+1}^{n} |\varphi_\nu(0)|^2}{\sum_{0}^{n} |\varphi_\nu(0)|^2} = \alpha$, let us have

$$-\ln \left(1 - \frac{\sum_{k+1}^{n} |\varphi_\nu(0)|^2}{\sum_{0}^{n} |\varphi_\nu(0)|^2} \right) = \sum_{s=1}^{\infty} \frac{\alpha^s}{s} \leqslant \alpha + \frac{\alpha^2}{1-\alpha} \leqslant$$
$$\leqslant \frac{2\alpha}{1-\alpha} \leqslant \alpha \cdot 2 \frac{\sum_{0}^{\infty} |\varphi_\nu(0)|^2}{|\varphi_0(0)|^2} = C_1 \alpha,$$

where through C_i here and below are designated constant. Hence and

from (4.22) follows the inequality

$$\ln \frac{D_s(|\varphi_n|^{-2})}{G(|\varphi_n|^{-2})^{s+1}} \leqslant \frac{C_1}{\sum\limits_0^n |\varphi_\nu(0)|^2} \sum_{k=0}^{n-1} \sum_{\nu=k+1}^{n} |\varphi_\nu(0)|^2 \leqslant$$
$$\leqslant \frac{C_1}{|\varphi_0(0)|^2} \sum_{\nu=1}^{\infty} \nu |\varphi_\nu(0)|^2 = C_2 \sum_{\nu=1}^{\infty} \nu |\varphi_\nu(0)|^2.$$

On lemma 11 then and

$$\iint\limits_{|z|\leqslant 1} \left| \frac{(\varphi_n^*(z))'}{\varphi_n^*(z)} \right|^2 d\sigma \leqslant C_3 \sum_{\nu=1}^{\infty} \nu |\varphi_\nu(0)|^2. \qquad (4.23)$$

On the strength of equality (4.16) of lemma 8 $\lim\limits_n \varphi_n^*(z) = \frac{1}{g_1(z)}$ it is evenly in any circle $|z| \leqslant r < 1$. Together with analytic functions $\varphi_n^*(z)$ in this circle evenly converge to $\left(\frac{1}{g_1(z)}\right)'$ also derivatives $(\varphi_n^*(z))'$. Therefore, by passing in inequality (4.23) to limit with n $\to \infty$, we will obtain on the basis of Fatu's lemma that

$$\iint\limits_{|z|\leqslant 1} \left| \frac{g_1'(z)}{g_1(z)} \right|^2 d\sigma \leqslant C_3 \sum_{\nu=1}^{\infty} \nu |\varphi_\nu(0)|^2. \qquad (4.24)$$

Is expressed now integral on the left side (4.24) through Fourier coefficients the summarized function $\ln g_1(e^{i\lambda})$.

Let us note first that the function $\ln g_1(e^{i\lambda})$ represents boundary value on circumference $|z| = 1$ function $\ln g_1(z)$, analytical in region $|z| < 1$ (external function $g_1(z)$ does not have as zeros in circle $|z| <$

1). Therefore the expansion of Fourier function $\ln g_1(e^{i\lambda})$ contains only nonnegative degrees $e^{i\lambda}$. Analogously Fourier series function $\ln \overline{g_1(e^{i\lambda})}$ contains only nonpositive degrees $e^{i\lambda}$. Furthermore,

$$\ln \overline{g_1(e^{i\lambda})} + \ln g_1(e^{i\lambda}) = \ln a(\lambda).$$

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Consequently, Fourier coefficients function $\ln g_1(e^{i\lambda})$ coincide with the appropriate Fourier coefficients function ln a (λ), i.e.,

$$\ln g_1(e^{i\lambda}) \sim \frac{1}{2} a_0 + \sum_{s=1}^{\infty} a_s e^{is\lambda}.$$

By passing to the calculation of integral in (4.24), let us note first that with $|z| \leqslant r < 1$

$$\frac{g_1'(z)}{g_1(z)} = (\ln g_1(z))' = \sum_{s=1}^{\infty} s a_s z^{s-1}.$$

Therefore

$$\iint\limits_{|z| \leqslant r} \left| \frac{g_1'(z)}{g_1(z)} \right|^2 d\sigma = \int_0^r \rho \, d\rho \int_{-\pi}^{\pi} \left| \sum_{s=1}^{\infty} s a_s \rho^{s-1} e^{i\lambda(s-1)} \right|^2 d\lambda =$$

$$= 2\pi \sum_{s=1}^{\infty} s^2 |a_s|^2 \int_0^r \rho^{2s-1} d\rho = \pi \sum_{s=1}^{\infty} s |a_s|^2 r^{2s} \to \pi \sum_{s=1}^{\infty} s |a_s|^2.$$

Hence and from inequality (4.24) follows the assertion of the theorem:

$$\sum_{s=-\infty}^{\infty} |s| |a_s|^2 < \infty.$$

Theorem 8 is completely demonstrated.

§5. Condition of absolute regularity. Processes with continuous time.

In the case of processes with continuous time we in state to demonstrate only the weaker (than theorem 8) result, analogous to lemma 6, toward formulation of which we now will begin.

Let ξ (t) be a stationary process with continuous time. Absolutely regular process is regular, so that without loss of generality it is possible to assume that the process ξ (t) has spectral densities f (λ) and that f (λ) = $|g(\lambda)|^2$, where $g \in \mathscr{H}^2$ in the upper half-plane.

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Let us designate by $c_\varepsilon(u)$ the Fourier transform functions $\frac{\overline{g(\lambda)}}{g(\lambda)} \frac{i}{i+\varepsilon\lambda}$.

Theorem 9. Stationary Gaussian process ξ (t) is absolutely regular in that and only that case, if it is regular for any T $\geqslant$ 0:

$$I(T) = \varlimsup_{\varepsilon \to 0} \int_{-\infty}^{-T} |u| \, |c_\varepsilon(u)|^2 \, du < \infty. \qquad (5.1)$$

Proof in many respects is analogous with the proof of lemma 6. Let us be based as before on theorem 4. In accordance with this theorem it suffices to establish that inequality I (T) < ∞ is equivalent to operator's nuclearity B_T. As in §4, will somewhat more conveniently to examine rassmatriat' the operators B_T^+, unitary equivalent to the operators B_T. If we, as in §4, do not vary designations with the standard isometric representation H (-∞, ∞) in L (F), then

$$B_T^+ = \mathcal{P}_T^+ \mathcal{P}_0^- \mathcal{P}_T^+,$$

where $\mathcal{P}_T^+$ - is a projector in L (F) on $L_{(T,\infty)}(F)$; $\mathcal{P}_0^-$ - projector in L (F) on $L_{(-\infty,0)}(F)$. It suffices to consider operator B_T^+ as operator from in $L_{(T,\infty)}(F)$ в $L_{(T,\infty)}(F)$.

Analogous with lemma 2 chapters II are proven the following equalities:

$$\mathcal{P}_T^+ = g^{-1}\Pi_T^+ g, \quad \mathcal{P}_0^- = \tilde{g}^{-1}\Pi_0^- \tilde{g}, \tag{5.2}$$

where to this once Π_T^+ - projector in $\mathcal{L}^2(-\infty,\infty)$ on $e^{i\lambda T}\mathcal{H}^2$, Π_0^- - projector in $\mathcal{L}^2(-\infty,\infty)$ on $\mathcal{H}^{2-}$.

Let us demonstrate the need for the conditions of theorem. Let us introduce the functions

$$e_x(\lambda;\varepsilon) = e_x(\lambda) = e_x = i\frac{e^{ix\lambda}}{g(\lambda)(i+\varepsilon\lambda)}, \quad x \geqslant 0.$$

It is obvious, $e_x \in L_{(x,\infty)}(F)$. (Recall that in accordance with results §2 chapter II $L_{(x,\infty)}(F) = \frac{e^{ix\lambda}}{\sigma}\mathcal{H}^2$.)

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Lemma 12. Integrals $S_T = \int\limits_T^\infty \langle B_T e_x, e_x\rangle_F\, dx < \infty$ in that and only that case, if $\int\limits_{-\infty}^{-T} |u|\,|c_e(u)|^2\, du < \infty$. Moreover,

$$S_T = \int\limits_{-\infty}^{-T} (|u| - T)\,|c_e(u)|^2\, du. \tag{5.3}$$

Proof. On the basis of formulas (5.2) for $x, y \geqslant T$

$$\langle B_T^+ e_x, e_y\rangle_F = \langle \mathcal{P}_0^- e_x, e_y\rangle_F =$$
$$= \int\limits_{-\infty}^{\infty} \pi_0^-\left(\frac{\overline{g(\lambda)}}{g(\lambda)}\,\frac{ie^{i\lambda x}}{i+\varepsilon\lambda}\right)\overline{\frac{\overline{g(\lambda)}}{g(\lambda)}\,\frac{ie^{iy\lambda}}{i+\varepsilon\lambda}}\, d\lambda.$$

Further,

$$\frac{\overline{g(\lambda)}}{g(\lambda)}\,\frac{ie^{ix\lambda}}{i+\varepsilon\lambda} = \int\limits_{-\infty}^{\infty} e^{iu\lambda} c_e(u-x)\, du,$$

$$\frac{\overline{g(\lambda)}}{g(\lambda)}\,\frac{ie^{iy\lambda}}{i+\varepsilon\lambda} = \int\limits_{-\infty}^{\infty} e^{iu\lambda} c_e(u-y)\, dy,$$

$$\Pi_0^-\,\frac{\bar{g}}{g}\,\frac{ie^{ix\lambda}}{1+\varepsilon\lambda} = \int\limits_{-\infty}^{0} e^{iu\lambda} c_e(u-x)\, du.$$

Consequently, for x, y $\geqslant$ T

$$\langle B_T^+ e_x, e_y \rangle_F = \int_{-\infty}^{0} c_\varepsilon (u-x)\, \overline{c_\varepsilon (u-y)}. \tag{5.4}$$

Specifically, if x $\geqslant$ T

$$\langle B_T^+ e_x, e_x \rangle_F = \int_{-\infty}^{-x} | c_\varepsilon (u) |^2 du, \tag{5.5}$$

and, which means,

$$\int_T^{\infty} \langle B_T^+ e_x, e_x \rangle_F \, dx = \int_T^{\infty} dx \int_{-\infty}^{-x} | c_\varepsilon (u) |^2 du. \tag{5.6}$$

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If integrals in (5.6) are final, function $C(x) = \int_{-\infty}^{-x} |c_e(u)|^2 du$ is integrated, but this function is monotonic, and therefore [1] with $x \to \infty$. $C(x) = o(x^{-1})$.

FOOTNOTE [1]. Let $C(x) \geqslant 0$, $C(x)\downarrow$ and $\int_T^{\infty} C(x)\, dx < \infty$. With large x

$$C(x) \leqslant \frac{2}{x} \int_{x/2}^{x} C(y)\, dy = o(x^{-1}).$$

ENDFOOTNOTE.

Therefore, by integrating integral in the right side of equality (5.6) in parts, let us find

$$\int_T^{\infty} dx \int_{-\infty}^{-x} |c_e(u)|^2 du = -TC(T) + \int_T^{\infty} u\,|c_e(-u)|^2 du =$$
$$= \int_{-\infty}^{-T} (|u| - T)\,|c_e(u)|^2 du.$$

Conversely, if descends integral in the right side of equality (5.3), then

$$C(x) \leqslant \frac{1}{|x|} \int_{-\infty}^{-x} |u||c_{\varepsilon}(u)|^2 du = o(x^{-1}),$$

and in (5.6) again it is possible to integrate in parts. Lemma is demonstrated.

Thus, let process ξ (t) is absolutely regular. According to theorem 4 this means that will be located the number τ, with $0 < \tau < \infty$, such, that the operator B_{τ}^{+} is a completely continuous operator with the final trace.

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Let $\varphi_l(\lambda) = \varphi_l$ - be operator's personal (calibrated) vectors B_{τ}^{+}, a μ_j - corresponding to them eigenvalues. With all $\varepsilon > 0$, $x \geqslant \tau$

$$\langle B_{\tau}^{+} e_x, e_x \rangle_F = \sum_l |\beta_l(x)|^2 \langle B_{\tau}^{+} \varphi_l, \varphi_l \rangle_F = \sum \mu_l |\beta_l(x)|^2, \quad (5.7)$$

where $\beta_l(x) = \langle e_x, \varphi_l \rangle_F$. Since

$$\varphi_l \in L_{(\tau, \infty)}(F) = \frac{e^{i\lambda\tau}}{g} \mathscr{H}^2,$$

functions φ_l can be registered in the form $\varphi_l = \frac{e^{i\lambda\tau}\psi_l}{g}$, $\psi_l \in \mathscr{H}^2$, so that

$$\beta_l(x) = \langle e_x, \varphi_l \rangle_F = i \int_{-\infty}^{\infty} \frac{e^{i\lambda(x-\tau)} \overline{\psi_l(\lambda)}}{i + \varepsilon\lambda} d\lambda,$$

$$\int_{-\infty}^{\infty} |\beta_l(x)|^2 dx = \int_{-\infty}^{\infty} \frac{|\psi_l(\lambda)|^2}{1 + \varepsilon^2\lambda^2} d\lambda = \left\| \frac{i\varphi_l}{1 + \varepsilon\lambda} \right\|_F^2.$$

Hence and from (5.7) follows the inequality

$$\int_{\tau}^{\infty} \langle B_{\tau}^{+} e_x, e_x \rangle_F \, dx = \sum_j \mu_j \left\| \frac{i\varphi_j}{i + \varepsilon\lambda} \right\|_F^2 \leqslant \leqslant \sum_j \mu_j \| \varphi_j \|_F^2 = \sum_j \mu_j = \operatorname{Sp} B_{\tau}^{+}.$$

On lemma 12

$$\overline{\lim_{\varepsilon \to 0}} \int_{-\infty}^{-\tau} (|u| - \tau) | c_{\varepsilon}(u) |^2 \, du = \overline{\lim_{\varepsilon \to 0}} \int_{\tau}^{\infty} \langle B_{\tau}^{+} e_x, e_x \rangle_F \, dx \leqslant \operatorname{Sp} B_{\tau}^{+}.$$

Let us place $T = 2\tau$. The last/latter inequality means that

$$\overline{\lim_{\varepsilon \to 0}} \int_{-\infty}^{-T} |u| | c_{\varepsilon}(u) |^2 \, du \leqslant \overline{\lim_{\varepsilon \to 0}} \, 2 \int_{-\infty}^{-\tau} (|u| - \tau) | c_{\varepsilon}(u) |^2 \, du \leqslant \leqslant \operatorname{Sp} B_{\tau}^{+} < \infty. \quad (5.8)$$

The need for the conditions of theorem is demonstrated.

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Let us demonstrate their sufficiency. We should construct any orthonormalized base $\{\alpha_j\}$ в $L_{(T,\infty)}(F)$ and to demonstrate that a series $\sum_j \langle B_T^+\alpha_j, \alpha_j\rangle_F$ descends. The sum of this series will be equal to operator's trace [1] B_T^+.

FOOTNOTE [1]. See [5], page 55; let us note that B_T^+ - the positive operators. ENDFOOTNOTE.

As this base we will select for the function

$$\alpha_j(\lambda) = e^{i\lambda T}\frac{a_j(\lambda)}{g(\lambda)}, \quad a_j(\lambda) = \frac{1}{\sqrt{\pi}}\frac{1}{i+\lambda}\left(\frac{i-\lambda}{i+\lambda}\right)^j,$$
$$j = 0, 1, \ldots$$

Lemma 13. Functions $\alpha_j(\lambda)$, $j = 0, 1, \ldots$, form hollow orthonormal system in $L_{(T,\infty)}(F)$.

Proof. Is sufficient to consider case of T = 0. Since $L_{(0,\infty)}(F) = L^{+}(F) = \frac{1}{g}\mathcal{H}^2$ (see §2 chapter II), it is to demonstrate that the functions $a_l(\lambda)$ form full/total/complete orthonormal system in $\mathcal{H}^2$.

Let for certainty $k \geqslant l$, then [2]

$$(a_k, a_l)^{(2)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+\lambda^2} \left(\frac{i-\lambda}{i+\lambda}\right)^{k-l} d\lambda.$$

FOOTNOTE [2]. Recall that $(\varphi, \psi)^{(2)}$ indicates the scalar product in hilbert space $\mathcal{L}^2(-\infty, \infty) \supseteq \mathcal{H}^2$. ENDFOOTNOTE.

Integrand in this integral is analytical in the upper half-plane, with the exception of pole at point i. The deduction of integrand at this point is equal to 0, if $k-l > 0$, and $1/2i$, if $k-l = 0$. Consequently,

$$(a_k, a_l)^{(2)} = \delta_{kl}.$$

The orthonormality of system $a_l, j = 0, 1, \ldots,$ and with it and system $a_l, j = 0, 1, \ldots,$ is establish/installed.

It remained to check the completeness of system $a_l, j = 0, 1, \ldots,$ in $\mathcal{H}^2$. Let $\varphi(\lambda) = \varphi \in \mathcal{H}^2$. It is necessary to demonstrate that as soon as $(\varphi, a_l)^{(2)} = 0$

for all j = 0, 1, ..., then $\phi(\lambda) \equiv 0$.

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The function $\varphi \in \mathscr{H}^2$ is the boundary value of function $\phi(z)$, $z = \lambda + i\mu$, analytical in upper half-plane $\mu > 0$ and such, that

$$\sup_{\mu} \int_{-\infty}^{\infty} |\varphi(\lambda + i\mu)|^2 \, d\lambda < \infty.$$

Therefore the integral

$$\int_{-\infty}^{\infty} \varphi(\lambda) \overline{a_j(\lambda)} \, d\lambda = (\varphi, a_j)^{(2)}$$

can be computed with the help of calculus of residues; integrand has in the upper half-plane the only pole of order j + 1 at point z = i. Consequently, for all j = 0, 1, ... they occur of the equality

$$\int_{-\infty}^{\infty} \varphi(\lambda) \overline{a_j(\lambda)} \, d\lambda = \frac{2\sqrt{\pi}}{j!} (-1)^{j+1} \frac{d^j}{dz^j} \varphi(z) (z+i)^{j+1} \Big|_{z=i} = 0.$$

Examining these equalities consecutively for j = 0, 1, ..., we will obtain that all derivatives $\varphi^{(j)}(i)$, $j = 0, 1, \ldots$ analytic functions $\phi(z)$ are equal to zero. Therefore $\phi(z) \equiv 0$, $\phi(\lambda) \equiv 0$. Lemma is demonstrated.

Let us pass to the calculation of sum $\sum_j \langle B_\tau^+ a_j, a_j \rangle_F$. Let us designate by $A_j(x)$ the transform the Fourier function $a_j(\lambda)$:

$$a_j(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{i\lambda x} A_j(x) \, dx.$$

Let, further, $\alpha_{je}(\lambda)=i\frac{\alpha_j(\lambda)}{i+\varepsilon\lambda}=\frac{i\cdot a_j(\lambda)}{1+\varepsilon\lambda}\frac{e^{i\lambda T}}{g(\lambda)}$. Then on the strength of (5.6)

$$\langle B_\tau^+\alpha_{j\varepsilon},\alpha_{j\varepsilon}\rangle_F=\int_0^\infty\int_0^\infty\langle B_T^+e_{x+\tau},e_{x+\tau}\rangle A_j(x)\overline{A_j(y)}\,dx\,dy=$$

$$=\int_{-\infty}^0du\left|\int_0^\infty c_\varepsilon(u-x-\tau)A_j(x)\,dx\right|^2.$$

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Relying on lemma 13, it is not difficult to show that the functions $A_j(x)$ form the full/total/complete orthonormal set in $\mathscr{L}^2(0,\infty)$. Actually, the Fourier transform of functions from $\mathscr{H}^2$ is converted into zero on half-line (-∞, 0) (neq) on the contrary, any function from $\mathscr{L}^2(0,\infty)$, if it are defined by zero on half-line (-∞, 0), can be considered as the Fourier transform of functions $\mathscr{H}^2$ (actually, this assertion is a known theorem of Peli - Weiner). Consequently, operator U: $\mathscr{H}^2\to\mathscr{L}^2(0,\infty)$, comparing each function from $\mathscr{H}^2$ its Fourier transform, it realizes isometric conformity between $\mathscr{H}^2$ and $\mathscr{L}^2(0,\infty)$. Specifically, since $\{a_j\}$ there is the full/total/complete orthonormal set in $\mathscr{H}^2$, function A_j form the full/total/complete orthonormal set in $\mathscr{L}^2(0,\infty)$ ∎.

FOOTNOTE [1]. It is not difficult to count, **that** $A_l(x)=e^{-x}L_l(2x)$, **where** $L_l(x)=$ $=\sum_{\nu=0}^{l} c_l^{\nu} \frac{(-x)^{\nu}}{\nu}$ **there** is Laguerre's j polynomial **(see [24]). ENDFOOTNOTE.**

Let us **examine function** $c_\varepsilon(u-x-T)$ **with fixed/recorded** u, T as cell/element **of space** $\mathscr{L}^2(0,\infty)$. **Let us register** $c_\varepsilon(u-x-T)$ **in** the form

$$c_\varepsilon(u-x-T)=\sum_{l=0}^{\infty}\gamma_l(u)\,A_l(x),$$

where

$$\gamma_l(u)=\int_0^{\infty}c_\varepsilon(u-x-T)\,A_l(x)\,dx$$

also, with all u

$$\sum_{l=0}^{\infty}|\gamma_l(u)|^2=\int_0^{\infty}|c_\varepsilon(u-x-T)|^2\,dx=\int_{-\infty}^{u-T}|c_\varepsilon(v)|^2\,dv.$$

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Consequently, with all $\varepsilon>0$

$$\sum_{l=0}^{\infty} \langle B_T \alpha_{l\varepsilon}, \alpha_{l\varepsilon} \rangle_F = \sum_{l=0}^{\infty} \int_{-\infty}^{0} du \left| \int_{0}^{\infty} c_\varepsilon (u - x - \tau) A_l(x)\, dx \right|^2 =$$

$$= \sum_{l=0}^{\infty} \int_{-\infty}^{0} |\gamma_l(u)|^2 du = \int_{-\infty}^{0} \int_{-\infty}^{u-T} |c_\varepsilon(v)|^2 dv =$$

$$= \int_{-\infty}^{-T} dx \int_{-\infty}^{x} |c_\varepsilon(u)|^2 du = \int_{-\infty}^{-T} (|u| - T) |c_\varepsilon(u)|^2 du.$$

By set/assuming $\varepsilon \to 0$, let us find that with all n

$$\sum_{l=0}^{n} \langle B_T \alpha_l, \alpha_l \rangle_F = \lim_{\varepsilon \to 0} \sum_{l=0}^{n} \langle B_T \alpha_{l\varepsilon}, \alpha_{l\varepsilon} \rangle_F =$$

$$= \varliminf_{\varepsilon \to 0} \sum_{l=0}^{n} \langle B_T \alpha_{l\varepsilon}, \alpha_{l\varepsilon} \rangle_F \leqslant \varliminf_{\varepsilon \to 0} \sum_{l=0}^{\infty} \langle B_T \alpha_{l\varepsilon}, \alpha_{l\varepsilon} \rangle_F =$$

$$= \varliminf_{\varepsilon \to 0} \int_{-\infty}^{-T} (|u| - T) |c_\varepsilon(u)|^2 du.$$

Therefore and

$$\mathrm{Sp}\,B_T = \sum_{l=0}^{\infty} \langle B_T \alpha_l, \alpha_l \rangle_F \leqslant \underline{\lim}_{\varepsilon \to 0} \int_{-\infty}^{-T} (|u| - T) |c_\varepsilon(u)|^2 du \leqslant$$
$$\leqslant \overline{\lim}_{\varepsilon \to 0} \int_{-\infty}^{-T} |u| |c_\varepsilon(u)|^2 du < \infty. \quad (5.9)$$

Theorem is demonstrated.

Observation. If we compare inequalities (5.8) and (5.9), it is not difficult to comprehend that in reality for all T there is, infinite possible, a limit $\lim_{\varepsilon \to 0} \int_{-\infty}^{-T} |u| |c_\varepsilon(u)|^2 du$ and that

$$\mathrm{Sp}\,B_T = \lim_{\varepsilon \to 0} \int_{-\infty}^{-T} (|u| - T) |c_\varepsilon(u)|^2 du.$$